大阪公立大 2021 サンプル

1 (1) (i) $n=1$ のとき.

$$S_2 = \sum_{k=1}^{2} \frac{(-1)^{k-1}}{k} = \frac{1}{1} - \frac{1}{2} = \frac{1}{2} \quad , \quad T_1 = \frac{1}{1+1} = \frac{1}{2}$$

よって $S_2 = T_1$ が成り立つ

(ii) $n = l$ のとき

$S_{2l} = T_l$ が成り立つと仮定する.

このとき.

$$S_{2l+2} = S_{2l} + \frac{1}{2l+1} - \frac{1}{2l+2} = \sum_{k=1}^{l} \frac{1}{l+k} + \frac{1}{2l+1} - \frac{1}{2l+2}$$

$$= \sum_{k=1}^{l-1} \frac{1}{l+k+1} + \frac{1}{l+1} + \frac{1}{2l+1} - \frac{1}{2l+2} = \sum_{k=1}^{l-1} \frac{1}{l+1+k} + \frac{1}{l+1+l} + \frac{1}{l+1+l+1}$$

$$= \sum_{k=1}^{l+1} \frac{1}{l+1+k} = T_{l+1}$$

よって仮定の下で $n = l+1$ でも $S_{2(l+1)} = T_{l+1}$ が成り立つ.

(i)(ii)より 全ての自然数 n に対して $S_{2n} = T_n$ が成り立つ

(2) $$\lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} T_n = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1+\frac{k}{n}} = \int_0^1 \frac{1}{1+x} dx = \left[\log|1+x| \right]_0^1 = \log 2$$

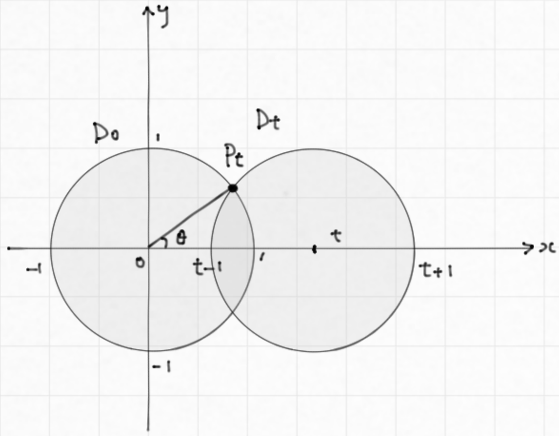
(3) $$\lim_{n\to\infty} S_{2n-1} = \lim_{n\to\infty} \left(S_{2n} + \frac{1}{2n} \right) = \log 2$$

2

問1　中心角 2θ, 半径 1 の扇形から頂角 2θ の二等辺三角形の面積を引いたものの 2倍.

$$S(t) = \left(\pi \cdot 1^2 \times \frac{2\theta}{2\pi} - \frac{1}{2} \times 1 \times 1 \times \sin 2\theta\right) \times 2$$

$$= 2\theta - \sin 2\theta$$



問2　P_t の x 座標は $\cos\theta$

対称性より. $\boldsymbol{t = 2\cos\theta}$

問3

$$\int_0^2 S(t)\,dt = \int_0^2 2\theta - \sin 2\theta \, dt$$

$t = 2\cos\theta$ だから $\dfrac{dt}{d\theta} = -2\sin\theta$,

t	$0 \to 2$
θ	$\frac{\pi}{2} \to 0$

$$\int_0^2 2\theta - \sin 2\theta \, dt = \int_{\frac{\pi}{2}}^{0} (2\theta - \sin 2\theta)(-2\sin\theta)\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} 4\theta\sin\theta - 4\sin^2\theta\cos\theta \, d\theta$$

$$= \left[-4\theta\cos\theta\right]_0^{\frac{\pi}{2}} + 4\int_0^{\frac{\pi}{2}} 1 \cdot \cos\theta \, d\theta - 4\left[\frac{1}{3}\sin^3\theta\right]_0^{\frac{\pi}{2}}$$

$$= -2\pi \times 0 - 0 + 4\left[\sin\theta\right]_0^{\frac{\pi}{2}} - 4\left(\frac{1}{3} - 0\right)$$

$$= 4 - 0 - \frac{4}{3} = \boldsymbol{\frac{8}{3}}$$

3

問1 $w = z + \frac{1}{z} = r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos(-\theta) + i\sin(-\theta))$

$= r\cos\theta + \frac{1}{r}\cos\theta + i\left(r\sin\theta - \frac{1}{r}\sin\theta\right)$

$u = \left(r+\frac{1}{r}\right)\cos\theta$, $v = \left(r-\frac{1}{r}\right)\sin\theta$

問2 $|z+1| = |z-i|$ の両辺2乗

$(z+1)(\bar{z}+1) = (z-i)(\bar{z}+i)$

$z\bar{z} + z + \bar{z} + 1 = z\bar{z} + zi - \bar{z}i + 1$

$z + \bar{z} = (z - \bar{z})i$

$z = r(\cos\theta + i\sin\theta)$ を代入

$2r\cos\theta = -2r\sin\theta$

$z \neq 0$ より $r \neq 0$ だから $\cos\theta = -\sin\theta$

$0 < \theta < \pi$ の範囲でこれを満たすのは $\theta = \frac{3}{4}\pi$

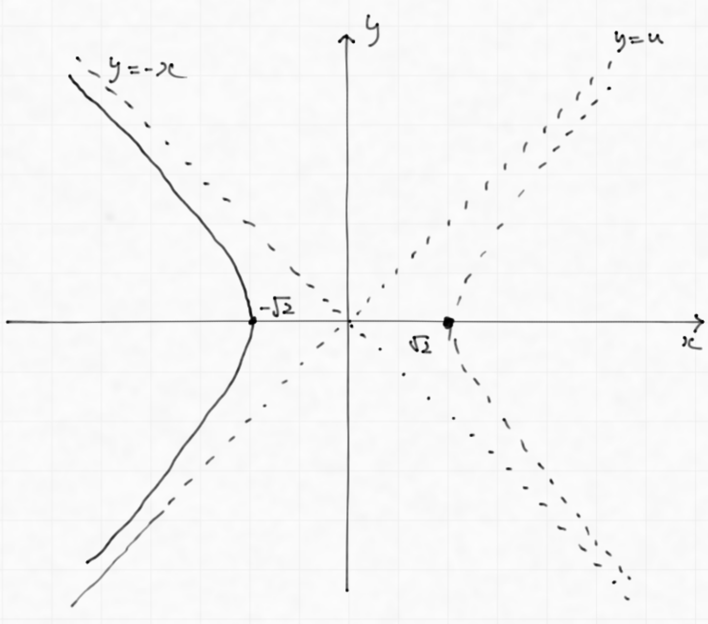
このとき、(1) より w は

$w = u + vi = \left(r+\frac{1}{r}\right)\times\cos\left(\frac{3}{4}\pi\right) + i\left(r-\frac{1}{r}\right)\sin\left(\frac{3}{4}\pi\right) = -\frac{1}{\sqrt{2}}\left(r+\frac{1}{r}\right) + \frac{1}{\sqrt{2}}i\left(r-\frac{1}{r}\right)$

$u + v = -\frac{\sqrt{2}}{r}$, $u - v = -\sqrt{2}r$

$(u+v)(u-v) = \left(-\frac{\sqrt{2}}{r}\right)(-\sqrt{2}r) = 2 \iff u^2 - v^2 = 2$.

また $r > 0$ より $u + v = -\frac{\sqrt{2}}{r} < 0$, $u - v = -\sqrt{2}r < 0$

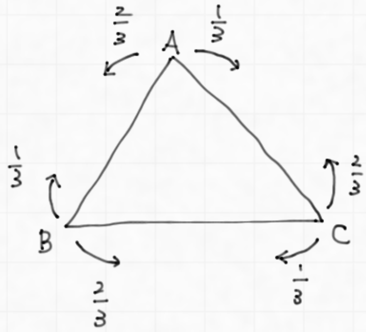


以上より w の軌跡は

$x^2 - y^2 = 2$, $y < -x$, $y > x$

で表される双曲線(左グラフ)

4



問1　右図より

$$a_{n+1} = \frac{1}{3}b_n + \frac{2}{3}c_n$$
$$b_{n+1} = \frac{1}{3}c_n + \frac{2}{3}a_n$$
$$c_{n+1} = \frac{1}{3}a_n + \frac{2}{3}b_n$$

問2

$$a_{n+2} = \frac{1}{3}b_{n+1} + \frac{2}{3}c_{n+1}$$
$$= \frac{1}{3}\left(\frac{1}{3}c_n + \frac{2}{3}a_n\right) + \frac{2}{3}\left(\frac{1}{3}a_n + \frac{2}{3}b_n\right)$$
$$= \frac{4}{9}a_n + \frac{4}{9}b_n + \frac{1}{9}c_n.$$

ここで、Pは、A,B,Cのいずれかに存在するので $a_n + b_n + c_n = 1$ が成り立つので、

$a_n + b_n = 1 - c_n$ を代入すると

$$a_{n+2} = \frac{4}{9}(1 - c_n) + \frac{1}{9}c_n = \frac{4}{9} - \frac{1}{3}c_n \qquad a_{n+2} = \frac{4}{9} - \frac{1}{3}c_n$$

問3　問2と同様に　$b_{n+2} = \frac{4}{9} - \frac{1}{3}a_n$，$c_{n+2} = \frac{4}{9} - \frac{1}{3}b_n$

$$a_{n+6} = \frac{4}{9} - \frac{1}{3}c_{n+4} = \frac{4}{9} - \frac{1}{3}\left(\frac{4}{9} - \frac{1}{3}b_{n+2}\right)$$
$$= \frac{8}{27} + \frac{1}{9}\left(\frac{4}{9} - \frac{1}{3}a_n\right) = \frac{28}{81} - \frac{1}{27}a_n \qquad a_{n+6} = \frac{28}{81} - \frac{1}{27}a_n$$

問4　$a_{n+6} = \frac{28}{81} - \frac{1}{27}a_n$　より

$a_{n+6} - \frac{1}{3} = -\frac{1}{27}\left(a_n - \frac{1}{3}\right)$　が成り立つ.

$$a_{6k+1} - \frac{1}{3} = -\frac{1}{27}\left(a_{6k-5} - \frac{1}{3}\right)$$
$$= \left(-\frac{1}{27}\right)^2\left(a_{6k-11} - \frac{1}{3}\right)$$
$$= \cdots = \left(-\frac{1}{27}\right)^k\left(a_1 - \frac{1}{3}\right) = \left(-\frac{1}{27}\right)^k\left(-\frac{1}{3}\right) \qquad (\because a_1 = 0)$$

$$a_{6k+1} = \frac{1}{3}\left(1 - \left(-\frac{1}{27}\right)^k\right)$$