山形大2018 工

1

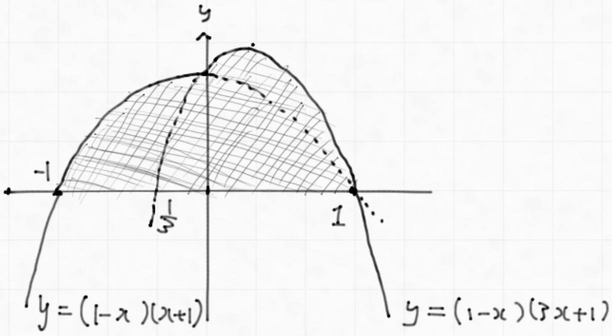
(1) $f'(x) = \dfrac{1}{\cos^2 x} - (\tan x)^{-2}(\tan x)' = \dfrac{1}{\cos^2 x} - \dfrac{1}{\sin^2 x} = \dfrac{2}{1+\cos 2x} - \dfrac{2}{1-\cos 2x}$

$f'\left(\dfrac{\pi}{8}\right) = \dfrac{2}{1+\frac{1}{\sqrt{2}}} - \dfrac{2}{1-\frac{1}{\sqrt{2}}} = \dfrac{2\sqrt{2}}{\sqrt{2}+1} - \dfrac{2\sqrt{2}}{\sqrt{2}-1} = 2\sqrt{2}\,\dfrac{\cancel{\sqrt{2}}-1-\cancel{\sqrt{2}}-1}{(\sqrt{2}+1)(\sqrt{2}-1)} = \mathbf{-4\sqrt{2}}$

(2) $x \geqq 0$ のとき　$y = (1-x)(3x+1)$ …①

$x < 0$ のとき　$y = (1-x)(x+1)$ …②

①②の交点は $x = 1, 0$



$S = \displaystyle\int_{-1}^{0} (1-x)(x+1)\,dx + \int_{0}^{1} (1-x)(3x+1)\,dx$

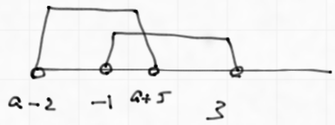
$= \displaystyle\int_{-1}^{0} 1-x^2\,dx + \int_{0}^{1} -3x^2+2x+1\,dx$

$= \left[x - \frac{1}{3}x^3\right]_{-1}^{0} + \left[-x^3+x^2+x\right]_{0}^{1} = -\left(-1+\frac{1}{3}\right) + (\cancel{-1}+\cancel{1}+1) = \mathbf{\dfrac{5}{3}}$

(3) $|x-1| \leqq 2 \iff -2 \leqq x-1 \leqq 2 \iff -1 \leqq x \leqq 3$ …①

$x^2 - (2a+3)x + (a+5)(a-2) \leqq 0 \iff (x-a+2)(x-a-5) \leqq 0$

$\iff a-2 \leqq x \leqq a+5$ …②



①かつ②が存在するための条件は　$a-2 \leqq 3$ かつ $a+5 \geqq -1$

すなわち、$a \leqq 5$ かつ $a \geqq -6$　　$\therefore\ \mathbf{-6 \leqq a \leqq 5}$

2

(1) $f'(x) = 1\cdot e^{-x} + x(-e^{-x}) = \mathbf{(1-x)e^{-x}}$

$f''(x) = -e^{-x} + (1-x)(-e^{-x}) = \mathbf{(x-2)e^{-x}}$

x	0	…	1	…
$f'(x)$		+	0	−
$f(x)$	0	↗	$\frac{1}{e}$	↘

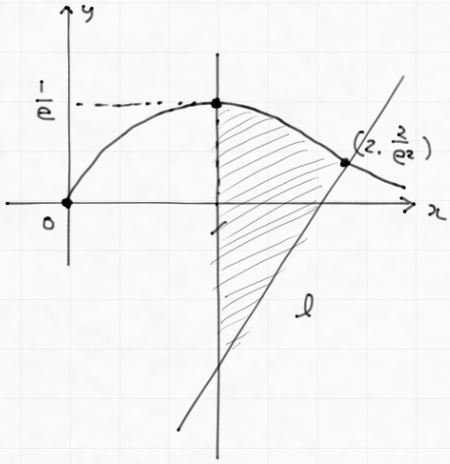
(2) $f'(x)=0$ となるのは $x=1$　$x>1$ のとき $f'(x)<0$ で

$f(x)$ の増減は右のとおり.

$f(0)=0$, $f(1)=\frac{1}{e}$, $\lim_{x\to\infty} f(x) = 0$

より. Cの概形は右下のようになる.

(変曲点は $f''(x)=0$ より $x=2$. $y=f(2)=\frac{2}{e^2}$)



(3) $f''(x)=0$ より $x=2$ のとき凹凸は切り替わる

$f'(2) = -e^{-2}$ だから 法線 l は

$$y = \frac{1}{e^{-2}}(x-2) + f(2)$$

$$\mathbf{y = e^2x - 2e^2 + \frac{2}{e^2}}$$

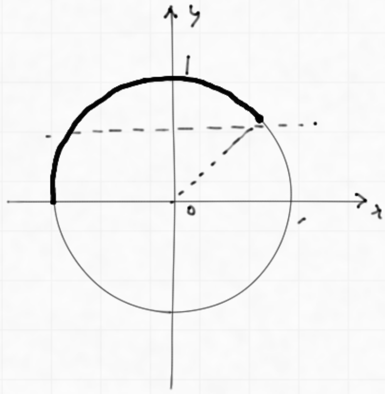
(4) $S = \int_1^2 xe^{-x} - (e^2x - 2e^2 + \frac{2}{e^2})\,dx$

$= \left[-xe^{-x} - e^{-x}\right]_1^2 - \left[\frac{1}{2}e^2x^2 - (2e^2 - \frac{2}{e^2})x\right]_1^2$

$= -2e^{-2} - e^{-2} - (-e^{-1} - e^{-1}) - \frac{1}{2}e^2\times 4 + (2e^2 - \frac{2}{e^2})\times 2 + (\frac{1}{2}e^2 - 2e^2 + \frac{2}{e^2})$

$= \frac{-2-1-4+2}{e^2} + \frac{1+1}{e} + \frac{1}{2}e^2 = \mathbf{\frac{1}{2}e^2 + \frac{2}{e} - \frac{5}{e^2}}$

3



(1) (i) $t=\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)+1$

$0\leqq x\leqq\frac{3}{4}\pi$ より　$\frac{\pi}{4}\leqq x+\frac{\pi}{4}\leqq\pi$

よって　$0\leqq\sin\left(x+\frac{\pi}{4}\right)\leqq 1$

$1\leqq\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)\leqq\sqrt{2}+1$　　∴ $1\leqq t\leqq\sqrt{2}+1$

(ii) $t-1=\sin x+\cos x$ の両辺を2乗して

$t^2-2t+1=1+2\sin x\cos x$

$2\sin x\cos x=t^2-2t$,　$\sin x+\cos x+1=t$ を C の式に代入

$y=t^2-2t-kt=t^2-(k+2)t$　　証明終

(2) $k=0$ のとき．$C: y=t^2-2t=0$ となるのは $t=0, 2$.

(i)より　$1\leqq t\leqq\sqrt{2}+1$ だから $t\neq 0$.

$t=2$ のとき　$\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)+1=2$　　$\sin\left(x+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$　　$x+\frac{\pi}{4}=\frac{\pi}{4}, \frac{3}{4}\pi$

∴ $x=0, \frac{1}{2}\pi$

$x=0$ のとき．$y=4-2\cdot 2=0$　　$(0, 0)$

$x=\frac{1}{2}\pi$ のとき　$y=0$.　　$\left(\frac{1}{2}\pi, 0\right)$　　**$x=0, \frac{1}{2}\pi$**

(3) $y=t^2-(k+2)t=t(t-k-2)$

$y=0$となるのは．$t=0, k+2$.

$1\leqq t\leqq\sqrt{2}+1$ より　$y=0$ となるのは　$1\leqq k+2\leqq\sqrt{2}+1$ が成り立つときで

(1)(i)より　$\frac{1}{\sqrt{2}}\leqq\sin\left(x+\frac{\pi}{4}\right)<1$ のとき．xは2つ存在する．

このとき　$1+1\leqq\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)+1=t<\sqrt{2}+1$ だから．

$2\leqq k+2<\sqrt{2}+1$ が成り立てば xは2つ存在する

∴ **$0\leqq k<\sqrt{2}-1$**

4

(1) $\int_0^{\frac{\pi}{2}} x\cos x\,dx = \left[x\sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x\,dx$

$= \frac{\pi}{2}\times 1 - 0\times 0 + \left[\cos x\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2} + 0 - 1 = \frac{\pi}{2} - 1$ 証明終

(2) $a_{n+2} = \int_0^{\frac{\pi}{2}} \left(a_{n+1}x + \frac{\pi}{2}a_n\right)\cos x\,dx$

$= a_{n+1}\int_0^{\frac{\pi}{2}} x\cos x\,dx + \frac{\pi}{2}a_n\int_0^{\frac{\pi}{2}}\cos x\,dx$

$= a_{n+1}\left(\frac{\pi}{2}-1\right) + \frac{\pi}{2}a_n(1-0)$ …①

$a_{n+2} + a_{n+1} = \frac{\pi}{2}(a_{n+1} + a_n)$

$\therefore$ $b_{n+1} = \frac{\pi}{2}b_n$ であり $\{b_n\}$ は公比 $\frac{\pi}{2}$ の等比数列

$b_1 = a_2 + a_1 = \pi + 1$ だから $b_n = (\pi+1)\left(\frac{\pi}{2}\right)^{n-1}$

(3) ① より

$a_{n+2} - \frac{\pi}{2}a_{n+1} = -\left(a_{n+1} - \frac{\pi}{2}a_n\right)$ $\therefore$ $c_{n+1} = -c_n$

$\{c_n\}$は公比 -1 の等比数列. $c_1 = a_2 - \frac{\pi}{2}a_1 = \frac{\pi}{2}$ $c_n = \frac{\pi}{2}(-1)^{n-1}$

(4) (2) (3) より $a_{n+2} + a_{n+1} = (\pi+1)\left(\frac{\pi}{2}\right)^{n-1}$ …③

$a_{n+2} - \frac{\pi}{2}a_{n+1} = \frac{\pi}{2}(-1)^{n-1}$ …④

③ − ④ $\left(1+\frac{\pi}{2}\right)a_{n+1} = (\pi+1)\left(\frac{\pi}{2}\right)^{n-1} - \frac{\pi}{2}(-1)^{n-1}$

$a_n = \dfrac{(\pi+1)\left(\frac{\pi}{2}\right)^{n-2} - \frac{\pi}{2}(-1)^{n-2}}{1+\frac{\pi}{2}} = \dfrac{2(\pi+1)\left(\frac{\pi}{2}\right)^{n-1} - \pi(-1)^{n-2}}{2+\pi}$

(5) $\lim_{n\to\infty}\dfrac{a_{n+1}}{a_n} = \lim_{n\to\infty}\dfrac{2(\pi+1)\left(\frac{\pi}{2}\right)^{n} - \pi(-1)^{n-1}}{2(\pi+1)\left(\frac{\pi}{2}\right)^{n-1} - \pi(-1)^{n-2}} = \lim_{n\to\infty}\dfrac{2(\pi+1)\times\frac{\pi}{2} - \pi\left(-\frac{2}{\pi}\right)^{n-1}}{2(\pi+1) + \pi\left(-\frac{2}{\pi}\right)^{n-1}} =$

$= \dfrac{2(\pi+1)}{2(\pi+1)}\times\frac{\pi}{2} = \frac{\pi}{2}$