関西大学2022

1 (1) 連立 $\frac{x^2}{4}+\frac{1}{9}(2x+k)^2=1 \iff 9x^2+4(4x^2+4kx+k^2)=36$

$\iff 25x^2+16kx+4k^2-36=0 \quad \cdots ①$

判別式をDとして $D<0$ のとき ① は解を持たず、だ円と直線は共有点を持たない

$D/4=(8k)^2-25(4k^2-36)=-36k^2+25\times36<0$

$k^2>25 \qquad \boldsymbol{k>5} \qquad (\because k$ は正の定数$)$

(2) Pを通って直線と垂直(傾き$-\frac{1}{2}$)な直線は

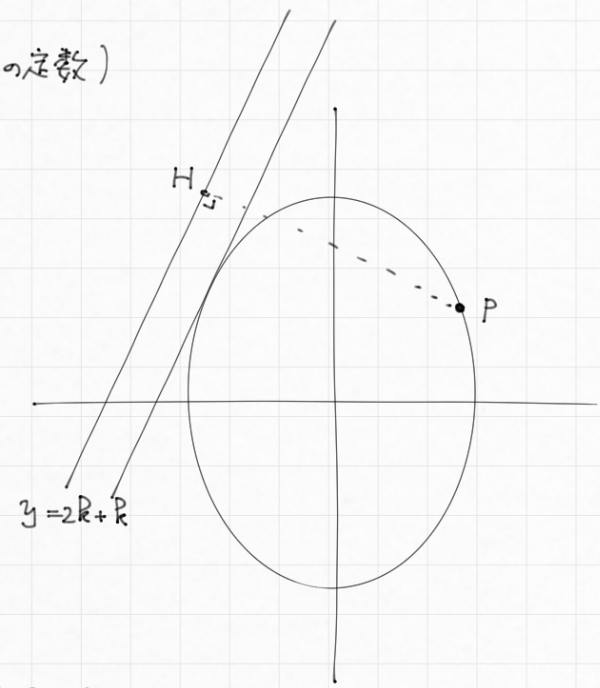
$y=-\frac{1}{2}(x-2\cos\theta)+3\sin\theta$

これと $y=2x+k$ の交点がH

$2x+k=-\frac{1}{2}x+\cos\theta+3\sin\theta$

$x=\frac{2}{5}(\cos\theta+3\sin\theta-k)$

$y=\frac{4}{5}(\cos\theta+3\sin\theta-k)+k$



$\boldsymbol{H\left(\frac{2}{5}(\cos\theta+3\sin\theta-k),\ \frac{4}{5}\cos\theta+\frac{12}{5}\sin\theta+\frac{1}{5}k\right)}$

(3) PHの長さが最も短かくなるのは だ円の接線の傾きが2となり、

かつ Pが第2象限にあるとき.

Pにおけるだ円の接線は $\frac{\cancel{2}\cos\theta\cdot x}{\cancel{4}\,2}+\frac{\cancel{3}\sin\theta\cdot y}{\cancel{9}\,3}=1$

その法線ベクトルは $\left(\frac{1}{2}\cos\theta,\ \frac{1}{3}\sin\theta\right)$

これが $(1,2)$ のベクトルと垂直 (直線 $y=2x+k$ と垂直だから)

$(1,2)\cdot\left(\frac{1}{2}\cos\theta,\ \frac{1}{3}\sin\theta\right)=\frac{1}{2}\cos\theta+\frac{2}{3}\sin\theta=\frac{1}{6}(3\cos\theta+4\sin\theta)$

$=\frac{\sqrt{5}}{6}\sin(\theta+\alpha)$ (ただし α は $\cos\alpha=\frac{4}{5},\ \sin\alpha=\frac{3}{5}$ を満たす角)

これが0となるとき、2つのベクトルは垂直だから.

$\sin(\theta+\alpha)=0 \qquad \therefore\ \theta=\pi-\alpha$

このとき $y=2x+k$ と P との距離 d は

ここから始めても大丈夫でした...

$d=\frac{|2\cdot2\cos\theta-3\sin\theta+k|}{\sqrt{2^2+(-1)^2}}=\frac{|4\cos(\pi-\alpha)-3\sin(\pi-\alpha)+k|}{\sqrt{5}}$

$=\frac{1}{\sqrt{5}}|-4\cos\alpha-3\sin\alpha+k|=\frac{1}{\sqrt{5}}\left|-\frac{16}{5}-\frac{9}{5}+k\right|=\frac{1}{\sqrt{5}}|k-5|=\boldsymbol{\frac{1}{\sqrt{5}}(k-5)}$

$(\because k>5)$

2

$$a_n = \int_1^3 x e^{nx^2} dx = \frac{1}{2n}\int_1^3 (nx^2)' e^{nx^2} dx = \frac{1}{2n}\Big[e^{nx^2} \Big]_1^3 = \frac{1}{2n}\left(e^{9n} - e^{n}\right)$$

$$\log n a_n = \log\left\{\frac{1}{2}\left(e^{9n} - e^{n}\right)\right\} = \log\left(e^{9n} - e^{n}\right) - \log 2$$

$$= \log e^{9n}\left(1 - e^{-8n}\right) - \log 2 = 9n + \log\left(1 - e^{-8n}\right) - \log 2$$

$e^{nx^2} \leqq x e^{nx^2} \leqq 3 e^{nx^2}$ より　　$\frac{1}{3} x e^{nx^2} \leqq e^{nx^2} \leqq x e^{nx^2}$

これを1〜3の範囲で積分

$$\frac{1}{3}\int_1^3 x e^{nx^2} dx \leqq \int_1^3 e^{nx^2} \leqq \int_1^3 x e^{nx^2} dx$$

$$\frac{1}{3} a_n \leqq b_n \leqq a_n$$

$$\frac{1}{3} n a_n \leqq n b_n \leqq n a_n$$

$$\log \frac{1}{3} n a_n \leqq \log n b_n \leqq \log n a_n$$

$$\frac{1}{n}\left(\log n a_n - \log 3\right) \leqq \frac{1}{n}\log n b_n \leqq \frac{1}{n}\log n a_n$$

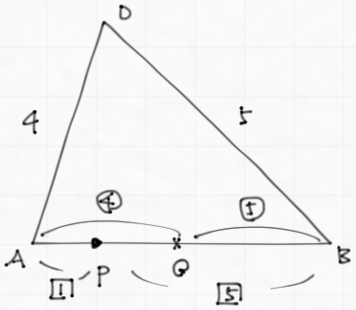
$$\lim_{n\to\infty} \frac{1}{n}\log n a_n = \lim_{n\to\infty} \frac{1}{n}\left(9n + \log\left(1 - e^{-8n}\right) - \log 2\right)$$

$$= \lim_{n\to\infty}\left(9 - \frac{1}{n}\log\left(1 - e^{-8n}\right) - \frac{\log 2}{n}\right) = 9 - 0 - 0 = 9$$

$$\lim_{n\to\infty} \frac{1}{n}\left(\log n a_n - \log 3\right) = 9$$

よって、はさみうちの原理より　$\lim_{n\to\infty} \frac{1}{n}\log n b_n = 9$

3



$\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ と表す.

(1) $|\overrightarrow{AB}|^2 = |\vec{b} - \vec{a}|^2 = |\vec{b}|^2 - 2\vec{a}\cdot\vec{b} + |\vec{a}|^2 = 25 - 5 + 16 = 36$

$|\overrightarrow{AB}| = \sqrt{36} = \mathbf{6}$

$16|\overrightarrow{OP}|^2 = |5\vec{a} + \vec{b}|^2 = 25\cdot16 + \overset{5}{\cancel{10}}\cdot\frac{5}{\cancel{2}} + 5^2 = 450$

$|\overrightarrow{OP}| = \frac{\sqrt{450}}{6} = \frac{\overset{5}{\cancel{15}}\sqrt{2}}{\underset{2}{\cancel{6}}} = \mathbf{\frac{5\sqrt{2}}{2}}$

(2) $\triangle OAB$の面積は $\frac{1}{2}\sqrt{|\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2} = \frac{1}{2}\sqrt{16+25-\frac{25}{4}} = \frac{15\sqrt{7}}{4}$

$\overrightarrow{PQ} = \overrightarrow{AQ} - \overrightarrow{AP} = \frac{4}{9}\overrightarrow{AB} - \frac{1}{6}\overrightarrow{AB} = \frac{8-3}{18}\overrightarrow{AB} = \frac{5}{18}\overrightarrow{AB}$

$\triangle OPQ = \frac{\overset{5}{\cancel{15}}\sqrt{7}}{4} \times \frac{5}{\underset{6}{\cancel{18}}} = \mathbf{\frac{25}{24}\sqrt{7}}$

(3) $|9\overrightarrow{OQ}|^2 = |5\vec{a} + 4\vec{b}|^2 = 25\times16 + 16\times5^2 + \overset{20}{\cancel{40}}\cdot\frac{5}{\cancel{2}} = 25(16+16+4) = 25\times36$

$|\overrightarrow{OQ}| = \frac{5\cdot6}{9} = \frac{10}{3}$

$\triangle OPQ = \frac{25}{24}\sqrt{7} = \frac{1}{2}|\overrightarrow{OP}||\overrightarrow{OQ}|\sin\angle POQ$

$\sin\angle POQ = \frac{\cancel{25}}{\underset{4}{\cancel{24}}}\sqrt{7} \times \cancel{2} \times \frac{\cancel{2}\cdot\cancel{3}}{\cancel{5}\sqrt{2}\cdot\cancel{10}} = \mathbf{\frac{\sqrt{14}}{8}}$

4

(1) 自然対数をとる　$x\log p = y\log q = z\log pq$

$$\frac{1}{x}+\frac{1}{y} = \frac{\log p}{z\log pq}+\frac{\log q}{z\log pq} = \frac{\log p+\log q}{z(\log p+\log q)} = \frac{1}{z}$$

(2) $$\frac{(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)^3}{\cos\gamma+i\sin\gamma} = \cos(\alpha+3\beta-\gamma)+i\sin(\alpha+3\beta-\gamma) \quad \cdots ①$$

$\alpha+3\beta-\gamma = \frac{\pi}{15}+\frac{3}{10}\pi-\frac{\pi}{5} = \frac{2\pi+9\pi-6\pi}{30} = \frac{1}{6}\pi$ だから ①は $\cos\frac{1}{6}\pi+i\sin\frac{1}{6}\pi = \frac{\sqrt{3}}{2}+\frac{1}{2}i$

(3) $2^a\cdot 3^b\cdot 6^c = 2^{a+c}\cdot 3^{b+c}$

その約数の個数は　$(a+c+1)(b+c+1) = 24$

$a\geqq 1,\ b\geqq 1,\ c\geqq 1$ に注意して.

$(a+c+1,\ b+c+1) = (3,8),(4,6),(6,4),(8,3)$

$(a+c+1,\ b+c+1) = (3,8)$ のとき.

$$\begin{cases} a+c+1=3 \\ b+c+1=8 \end{cases} \iff \begin{matrix} a+c=2 \\ b+c=7 \end{matrix} \qquad a=c=1,\ b=6$$

$(a+c+1,\ b+c+1) = (4,6)$ のとき

$$\begin{cases} a+c=3 \\ b+c=5 \end{cases} \qquad \begin{matrix} c=1\ \ a=2\ \ b=4 \\ c=2,\ a=1,\ b=3 \end{matrix}$$

以下同様

$(a,b,c) = (1,6,1),(2,4,1),(1,3,1),(6,1,1),(4,2,1),(3,1,1)$　6通り

$\therefore \frac{6}{6^3} = \frac{1}{36}$

(4) $f'(x) = \frac{\frac{1}{x}\cdot x-\log x}{x^2} = \frac{1-\log x}{x^2}$　　$f'(x)=0$ となるのは $x=e$

x	0	…	e	…
$f'(x)$		+	0	−
$f(x)$		↗		↘

$f(x)$ は $x=e$ で最大　最大値は $\frac{1}{e}$

(5) $$S_n = \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^2 = \frac{1}{n^3}\sum_{k=1}^{n}k^2 = \frac{1}{n^3}\times\frac{1}{6}n(n+1)(2n+1) = \frac{1}{6n^2}(2n^2+3n+1) = \frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}$$

$$S = \int_0^1 x^2\,dx = \frac{1}{3}$$

$$S_n - S = \frac{1}{2n}+\frac{1}{6n^2}$$

$n=49$　$\frac{1}{98}+\frac{1}{6\cdot 49^2} > \frac{1}{100}$　　$n=50$　$\frac{1}{100}+\frac{1}{60000} > \frac{1}{100}$

$n=51$　$\frac{1}{102}+\frac{1}{6\cdot 51^2} < \frac{1}{100}$　　　$n=51$