日本医科大学前期 2022

1 $|z| = 2$

$$w = \frac{(4+2i)z+4-4i}{z+2-2i} \quad \cdots ① \quad \Leftrightarrow \quad wz+2w-2wi = 4z+2iz+4-4i$$

$$\Leftrightarrow \quad (w-4-2i)z = -2w+2iw+4-4i$$

$w = 4+2i$ のとき、上式は成り立たないので $w \neq 4+2i$

このとき $z = \dfrac{-2w+2iw+4-4i}{w-4-2i}$

これを $|z|=2$ に代入

$$|(-2+2i)(w-2)| = 2\,|w-4-2i|$$

両辺2乗

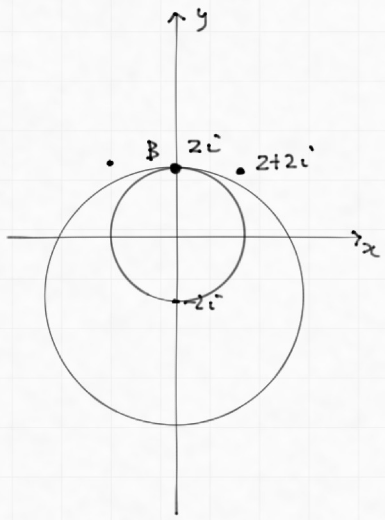
$$2\cancel{8}(w\bar{w}-2w-2\bar{w}+4) = \cancel{4}(w-4-2i)(\bar{w}-4+2i)$$

$$w\bar{w}-2iw+2i\bar{w}-12=0$$

$$(w+2i)(\bar{w}-2i) = 16$$

$$|w+2i| = 4$$

中心は $\alpha = -2i$, 半径 4



右上図より $z = w$ となる可能性のあるのは $z = w = 2i$ のみでこれは ① を満たしている

$\therefore \beta = 2i$

$$\frac{z-w}{z-\beta} = \frac{z - \dfrac{(4+2i)z-4-4i}{z+2-2i}}{z-2i} = \frac{z^2-2z-4iz+4+4i}{(z-2i)(z+2-2i)} = \frac{(z-2i)(z-2i-2)}{(z-2i)(z+2-2i)}$$

$$= \frac{z-2-2i}{z+2-2i}$$

$\sqrt{5}\,|w-z| \leqq |z-\beta|$ より $\dfrac{|z-w|}{|z-\beta|} \leqq \dfrac{1}{\sqrt{5}}$

ここに先の結果を代入 $\dfrac{|z-2-2i|}{|z+2-2i|} \leqq \dfrac{1}{\sqrt{5}}$

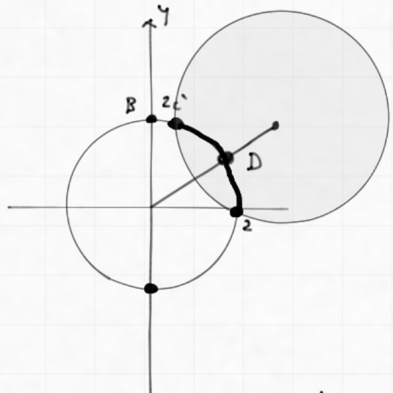
両辺2乗 $5(z\bar{z}-(2+2i)\bar{z}-(2-2i)z+8) \leqq z\bar{z}+(2-2i)\bar{z}+(2+2i)z+8$

$$4z\bar{z}-(12+8i)\bar{z}-(12-8i)z+32 \leqq 0$$

$$z\bar{z}-(3-2i)z-(3+2i)\bar{z}+8 \leqq 0$$

$$(z-3-2i)(\bar{z}-3+2i) \leqq 5$$

$$|z-3-2i| \leqq \sqrt{5}$$



C: $x^2+y^2=4$ と上の円の周 $(x-3)^2+(y-2)^2=5$

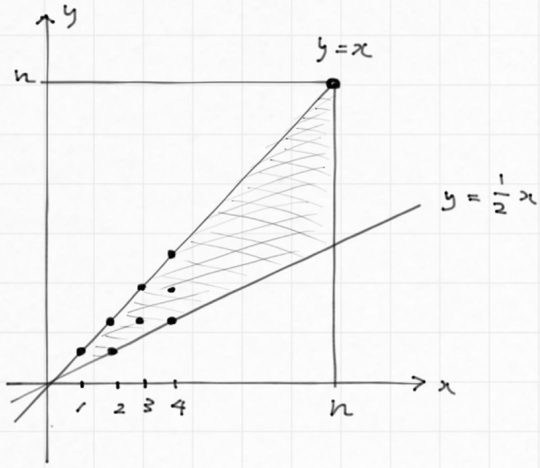
との交点は、$-6x-4y=-12$ $\quad 3x+2y=6$

これと $x^2+y^2=4$ を連立、$y=0, \frac{24}{13}$ $\quad (x, y) = (2, 0), \left(\frac{24}{13}, \frac{10}{13}\right)$ これとB $(0, 2)$ との距離をもとめて

最大値は $z=2$ のときで $\mathbf{2\sqrt{2}}$, 最小値は $z = \frac{24}{13}+\frac{10}{13}i$ のときで $\mathbf{\dfrac{2\sqrt{26}}{13}}$

2

(1) Aから赤玉 $= \frac{1}{6} \times \frac{x}{n}$　　Bから赤玉 $= \frac{5}{6} \times \frac{y}{n}$



$$p = \frac{\frac{x}{6n}}{\frac{x}{6n} + \frac{5y}{6n}} = \frac{x}{x+5y}$$

(2) $\frac{1}{6} \leqq \frac{x}{x+5y} \leqq \frac{2}{7}$　$\Leftrightarrow$　$y \leqq x \leqq 2y$

$n = 2m$ のとき.

$$N(n) = 1+2+2+3+3+\cdots+m+m+(m+1)$$

$$= m(m+1)+m = m(m+2) = \frac{1}{2}n\left(\frac{1}{2}n+2\right) = \frac{1}{4}n(n+4)$$

$n = 2m-1$ のとき.

$$N(n) = 1+2+2+3+3+\cdots+m+m$$

$$= m(m+1)-1 = \frac{n+1}{2} \times \frac{n+3}{2} - 1 = \frac{1}{4}n^2 + n - \frac{1}{4}$$

$$N(n) = \begin{cases} \frac{1}{4}n^2+n & (n\text{が偶数}) \\ \frac{1}{4}n^2+n-\frac{1}{4} & (n\text{が奇数}) \end{cases}$$

(3) $\frac{1}{4}n^2+n$ について

$n=90$ のとき　$\frac{1}{4}\times 90^2+90 = 2025+90 > 2020$

$n=88$ のとき　$\frac{1}{4}\times 88^2+88 = 88\times 23 = 2024 > 2020$

$n=86$　〃　$\frac{1}{4}\times 86^2+86 = 86\times 22.5 < 2020$

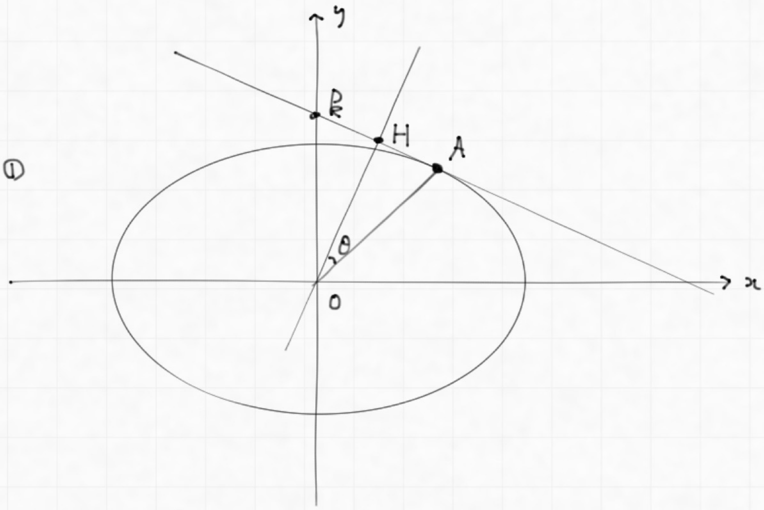
$\frac{1}{4}n^2+n-\frac{1}{4}$ について.

$n=89$ のとき　$\frac{1}{4}\times 89^2+89-\frac{1}{4} > 2020$

$n=87$ のとき　$\frac{1}{4}\times 87^2+87-\frac{1}{4} = 1979 < 2020$

以上より $n = 87$

3



(1) 連立 $\dfrac{x^2}{a^2}+\dfrac{1}{b^2}(m^2x^2-2mkx+k^2)=1$

$(b^2+a^2m^2)x^2-2a^2mkx+a^2k^2-a^2b^2=0 \cdots ①$

判別式をDとして

$D/4 = a^4m^2k^2-(b^2+a^2m^2)(a^2k^2-a^2b^2)=0$

$-a^2b^2k^2+a^2b^4+a^4b^2m^2=0$

$k^2=b^2+a^2m^2$

このとき①は $k^2x^2-2a^2mkx+a^4m^2=0$

$(kx-a^2m)^2=0$

$x=\dfrac{a^2m}{k}=\dfrac{a^2m}{\sqrt{b^2+a^2m^2}}$

$y=\dfrac{-a^2m^2}{\sqrt{b^2+a^2m^2}}+\sqrt{b^2+a^2m^2}=\dfrac{b^2}{\sqrt{b^2+a^2m^2}}$

$A\left(\dfrac{a^2m}{\sqrt{b^2+a^2m^2}},\ \dfrac{b^2}{\sqrt{b^2+a^2m^2}}\right)$

(2) ℓ: $mx+y-k=0$ と $O(0,0)$ との距離

$OH=\dfrac{|k|}{\sqrt{m^2+1}}=\dfrac{\sqrt{b^2+a^2m^2}}{\sqrt{m^2+1}}$

(3) $OA=\dfrac{1}{\sqrt{b^2+a^2m^2}}\sqrt{a^4m^2+b^4}$

$\sin\theta=\sqrt{1-\cos^2\theta}=\sqrt{1-\left(\dfrac{OH}{OA}\right)^2}=\sqrt{1-\dfrac{(b^2+a^2m^2)^2}{(a^4m^2+b^4)(m^2+1)}}$

$=\sqrt{\dfrac{m^2(a^4-2a^2b^2+b^4)}{(a^4m^2+b^4)(m^2+1)}}=\dfrac{m(a^2-b^2)}{\sqrt{(a^4m^2+b^4)(m^2+1)}}$

(4) $\sin\theta=\dfrac{(a^2-b^2)}{\sqrt{(a^4m^2+b^4)(1+\frac{1}{m^2})}}$

分母の中身 $=a^4m^2+b^4\cdot\dfrac{1}{m^2}+b^4+a^4 \geq a^4+b^4\ 2\sqrt{a^4b^4}=(a^2+b^2)^2$

$\sin\theta \leq \dfrac{a^2-b^2}{a^2+b^2}=M(a,b)$ 等号は $a^4m^2=\dfrac{b^4}{m^2} \Leftrightarrow am=b$ のとき.

(5) $\dfrac{a}{b}=t$ とおく $a>b$ だから $t>1$

$M=\dfrac{(\frac{a}{b})^2-1}{(\frac{a}{b})^2+1}=\dfrac{t^2-1}{t^2+1}=1-\dfrac{2}{t^2+1}$

$a=bt$ を条件に代入 $b^2t^2-2b^2t+b^2+b^2-2b+1 \leq \dfrac{3}{4}$

$(t^2-2t+2)b^2-2b+\dfrac{1}{4} \leq 0$

$b>0$ の範囲で b が存在するための条件は $t^2-2t+2=(t-1)^2+1>0$ だから
判別式を D_2 として $D_2 \geqq 0$.

$$D_2/4 = 1-\frac{1}{4}t^2+\frac{1}{2}t-\frac{1}{2} \geqq 0$$

$$t^2-2t-2 \leqq 0$$

$$1-\sqrt{3} \leqq t \leqq 1+\sqrt{3}$$

$t>1$ と併せて $1<t \leqq 1+\sqrt{3}$.

したがって $M = 1-\dfrac{2}{t^2+1} \leqq 1-\dfrac{2}{(1+\sqrt{3})^2+1} = \dfrac{3+4\sqrt{3}}{13}$

4

(1) $f'(x) = \frac{1}{2}\sqrt{\frac{1+\sqrt{x}}{1-\sqrt{x}}} \times \frac{-\frac{1}{2\sqrt{x}}(1+\sqrt{x}) - (1-\sqrt{x}) \times \frac{1}{2\sqrt{x}}}{(1+\sqrt{x})^2}$

$$= -\frac{1}{2\sqrt{x}}\sqrt{\frac{1+\sqrt{x}}{1-\sqrt{x}}} \times \frac{1}{(1+\sqrt{x})^2} = -\frac{1}{2\sqrt{x}} \times \frac{\sqrt{1-x}}{1-\sqrt{x}} \times \frac{1}{(1+\sqrt{x})^2}$$

$$= -\frac{1}{2\sqrt{x}} \times \frac{1}{1-x} \times \frac{\sqrt{1-x}}{1+\sqrt{x}} = -\frac{1}{2\sqrt{x}(1-x)} \times \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$$

よって　$0<x<1$ のとき $f'(x)<0$

$$f''(x) = \frac{2\times\frac{1}{2}\frac{1}{\sqrt{x}}(1-x)+2\sqrt{x}(-1)}{\{2\sqrt{x}(1-x)\}^2} \times \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} - \frac{1}{2\sqrt{x}(1-x)} \times \left(-\frac{1}{2\sqrt{x}(1-x)}\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\right)$$

$$= \frac{\frac{1}{\sqrt{x}}-\sqrt{x}-2\sqrt{x}}{4x(1-x)^2}\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} + \frac{1}{4x(1-x)^2}\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$$

$$= \frac{1}{4x(1-x)^2\sqrt{x}}\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\left(-3x+\sqrt{x}+1\right)$$

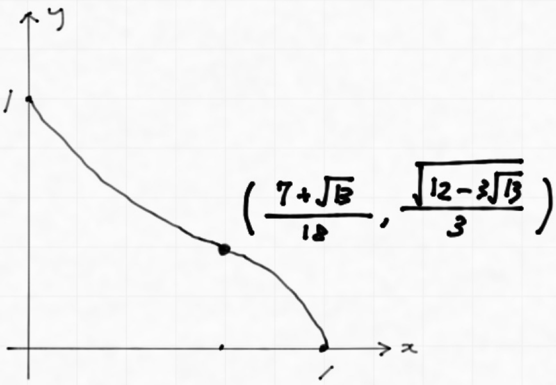
$-3x+\sqrt{x}+1=0$ となるのは　$\sqrt{x} = \frac{-1\pm\sqrt{1+12}}{-6} = \frac{1\mp\sqrt{13}}{6}$

$0<x<1$ を満たすのは　$x = \frac{\sqrt{13}+1}{6}$　　このとき $x = \left(\frac{\sqrt{13}+1}{6}\right)^2 = \frac{7+\sqrt{13}}{18}$

増減表は下のようになる

x	0	$\cdots$	$\frac{7+\sqrt{13}}{18}$	$\cdots$	1
$f'(x)$		$-$		$-$	
$f''(x)$		$+$	0	$-$	
$f(x)$	1	↘		↘	0

$$f\left(\frac{7+\sqrt{13}}{18}\right) = \sqrt{\frac{1-\frac{\sqrt{13}+1}{6}}{1+\frac{\sqrt{13}+1}{6}}} = \sqrt{\frac{5-\sqrt{13}}{7+\sqrt{13}}} = \sqrt{\frac{\cancel{48}^{12}-\cancel{12}^{3}\sqrt{13}}{\cancel{36}_{9}}} = \frac{\sqrt{12-3\sqrt{13}}}{3}$$



$$V = \int_0^1 \pi x^2 \, dy$$

$y = \sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}$ より　$y^2 = \dfrac{1-\sqrt{x}}{1+\sqrt{x}}$　$\Leftrightarrow$　$\sqrt{x} = \dfrac{1-y^2}{1+y^2} = -1 + \dfrac{2}{1+y^2}$

だから

$$V = \pi \int_0^1 \left(\frac{2}{1+y^2} - 1 \right)^4 dy$$

$y = \tan\theta$ とおく.　$\dfrac{dy}{d\theta} = \dfrac{1}{\cos^2\theta}$

y	$0 \to 1$
θ	$0 \to \frac{\pi}{4}$

$$V = \pi \int_0^{\frac{\pi}{4}} (2\cos^2\theta - 1)^4 \frac{1}{\cos^2\theta} \, d\theta$$

$$= \pi \int_0^{\frac{\pi}{4}} 16\cos^6\theta - 32\cos^4\theta + 24\cos^2\theta - 8 + \frac{1}{\cos^2\theta} \, d\theta$$

$$\int_0^{\frac{\pi}{4}} \cos^6\theta \, d\theta = \int_0^{\frac{\pi}{4}} \left(\frac{1+\cos 2\theta}{2} \right)^3 d\theta = \frac{1}{8} \int_0^{\frac{\pi}{4}} 1 + 3\cos 2\theta + \frac{3}{2}(1+\cos 4\theta) + (1-\sin^2 2\theta)\cos 2\theta \, d\theta$$

$$= \frac{1}{8} \left[\theta + \frac{3}{2}\sin 2\theta + \frac{3}{2}\theta + \frac{3}{8}\sin 4\theta + \frac{1}{2}\sin 2\theta - \frac{1}{6}\sin^3 2\theta \right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{8} \left(\frac{5}{8}\pi + \frac{3}{2} + 0 + \frac{1}{2} - \frac{1}{6} \right) = \frac{5}{64}\pi + \frac{11}{48}$$

$$\int_0^{\frac{\pi}{4}} \cos^4\theta \, d\theta = \frac{1}{4} \int_0^{\frac{\pi}{4}} 1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2} \, d\theta = \frac{1}{4} \left[\theta + \sin 2\theta + \frac{1}{2}\theta + \frac{1}{8}\sin 4\theta \right]_0^{\frac{\pi}{4}}$$

$$= \frac{3}{32}\pi + \frac{1}{4} + 0 = \frac{3}{32}\pi + \frac{1}{4}$$

$$\int_0^{\frac{\pi}{4}} \cos^2\theta \, d\theta = \frac{1}{2} \int_0^{\frac{\pi}{4}} 1 + \cos 2\theta \, d\theta = \frac{1}{2} \left[\theta + \frac{1}{2}\sin 2\theta \right]_0^{\frac{\pi}{4}} = \frac{\pi}{8} + \frac{1}{4}$$

$$\int_0^{\frac{\pi}{4}} \frac{1}{\cos^2\theta} \, d\theta = \left[\tan\theta \right]_0^{\frac{\pi}{4}} = 1$$

$$V = \pi \left\{ 16\left(\frac{5}{64}\pi + \frac{11}{48} \right) - 32\left(\frac{3}{32}\pi + \frac{1}{4} \right) + 24\left(\frac{\pi}{8} + \frac{1}{4} \right) - 8 \times \frac{\pi}{4} + 1 \right\}$$

$$= \pi \left(\frac{5}{4}\pi + \frac{11}{3} \cancel{- 3\pi} - 8 \cancel{+ 3\pi} + 6 - 2\pi + 1 \right)$$

$$= \pi \left(-\frac{3}{4}\pi + \frac{8}{3} \right) = \boldsymbol{\frac{8}{3}\pi - \frac{3}{4}\pi^2}$$