山形大学2019　工学部

1

(1) $$\sum_{n=1}^{\infty}\frac{1}{n(n+2)} = \lim_{N\to\infty}\sum_{n=1}^{N}\left(\frac{1}{n}-\frac{1}{n+2}\right)\times\frac{1}{2}$$

$$= \lim_{N\to\infty}\left\{\left(\frac{1}{1}-\cancel{\frac{1}{3}}\right)\times\frac{1}{2}+\left(\frac{1}{2}-\cancel{\frac{1}{4}}\right)\times\frac{1}{2}+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{5}}\right)\times\frac{1}{2}+\cdots+\left(\cancel{\frac{1}{N-2}}-\cancel{\frac{1}{N}}\right)\times\frac{1}{2}+\left(\cancel{\frac{1}{N-1}}-\frac{1}{N+1}\right)\times\frac{1}{2}+\left(\cancel{\frac{1}{N}}-\frac{1}{N+2}\right)\times\frac{1}{2}\right\}$$

$$= \lim_{N\to\infty}\left(\frac{1}{2}+\frac{1}{4}-\frac{1}{N+1}-\frac{1}{N+2}\right)\times\frac{1}{2} = \mathbf{\frac{3}{4}}$$

(2) $$\left(\frac{\sqrt{3}+i}{2}\right)^{-27} = \left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^{-27} = \cos\left(-\frac{9}{2}\pi\right)+i\sin\left(-\frac{9}{2}\pi\right) = \cos\frac{3}{2}\pi+i\sin\frac{3}{2}\pi = \mathbf{-i}$$

(3) 両辺を$3mn$倍

$$3(m+1)(n+1) = 4mn$$

$$3mn+3m+3n+3 = 4mn$$

$$mn-3m-3n-3 = 0$$

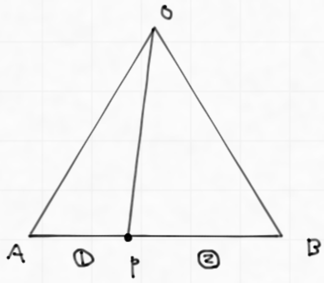
$$(m-3)(n-3) = 12 = 2^2\cdot 3^1$$

$m-3 < n-3$　および　$m-3 \geqq 1-3 = -2$　より　上式を満たすのは.

$$(m-3,\ n-3) = (1,\ 12),\ (2,\ 6),\ (3,\ 4)$$

$$(m,\ n) = \mathbf{(4,\ 15),\ (5,\ 9),\ (6,\ 7)}$$

2



(1) $|\overrightarrow{OA}| = 1.\ |\overrightarrow{OB}| = 1$

$\overrightarrow{OA}\cdot\overrightarrow{OB} = |\overrightarrow{OA}||\overrightarrow{OB}|\cos 60^\circ = \frac{1}{2}$

(2) $\overrightarrow{OP} = \frac{2}{3}\overrightarrow{OA} + \frac{1}{3}\overrightarrow{OB}$

$\overrightarrow{OQ} = t\,\overrightarrow{OP} = \frac{2}{3}t\,\overrightarrow{OA} + \frac{1}{3}t\,\overrightarrow{OB}$

よって $\overrightarrow{AQ} = \overrightarrow{OQ} - \overrightarrow{OA} = (\frac{2}{3}t - 1)\overrightarrow{OA} + \frac{1}{3}t\,\overrightarrow{OB}$

$\overrightarrow{BQ} = \overrightarrow{OQ} - \overrightarrow{OB} = \frac{2}{3}t\,\overrightarrow{OA} + (\frac{1}{3}t - 1)\overrightarrow{OB}$ 証明終

(3) (i) $S = |\overrightarrow{OQ}|^2 + |\overrightarrow{AQ}|^2 + |\overrightarrow{BQ}|^2$

$= \frac{1}{9}\left\{|2t\overrightarrow{OA} + t\overrightarrow{OB}|^2 + |(2t-3)\overrightarrow{OA} + t\overrightarrow{OB}|^2 + |2t\overrightarrow{OA} + (t-3)\overrightarrow{OB}|^2\right\}$

$= \frac{1}{9}\left(4t^2 + t^2 + 2t^2 + (2t-3)^2 + t^2 + t(2t-3) + 4t^2 + (t-3)^2 + 2t(t-3)\right)$

$= \frac{1}{9}\left(21t^2 - 27t + 18\right) = \frac{7}{3}t^2 - 3t + 2$

(ii) $S = \frac{7}{3}\left(t - \frac{9}{14}\right)^2 + \frac{29}{28}$

$0 \leqq t \leqq 1$だから $t = \frac{9}{14}$のとき Sは最小. 最小値は $\frac{29}{28}$

3

(1) $3a_{n+2}-4a_{n+1}+a_n=4+\frac{1}{2^{n+1}}$ $(n=1,2,3\dots)$

$3(a_{n+2}-a_{n+1})-(a_{n+1}-a_n)=4+\frac{1}{2^{n+1}}$ $(n=1,2,3\dots)$

$3b_{n+1}-b_n=\mathbf{4+\frac{1}{2^{n+1}}}$ $(n=1,2,3\dots)$

(2) (1)の結果に$b_n=c_n+\frac{1}{2^n}+2$を代入

$3\left(c_{n+1}+\frac{1}{2^{n+1}}+2\right)-c_n-\frac{1}{2^n}-2=4+\frac{1}{2^{n+1}}$

$3c_{n+1}-c_n+\frac{1}{2^{n+1}}(3-2)+4=4+\frac{1}{2^{n+1}}$

$3c_{n+1}-c_n=0$

$c_{n+1}=\frac{1}{3}c_n$ $(n=1,2,3\dots)$

これは$\{c_n\}$が公比$\frac{1}{3}$の等比数列であることを示している

(3) $c_1=b_1-\frac{1}{2^1}-2$

$b_1=a_2-a_1=\frac{17}{6}$ だから $c_1=\frac{17}{6}-\frac{1}{2}-2=\frac{1}{3}$

$c_n=c_1\times\left(\frac{1}{3}\right)^{n-1}=\left(\frac{1}{3}\right)^n$

$b_n=c_n+\frac{1}{2^n}+2=\mathbf{\frac{1}{3^n}+\frac{1}{2^n}+2}$ $(n=1,2,3\dots)$

(4) $\{b_n\}$は$\{a_n\}$の階差数列だから

$n\geqq 2$のとき、

$a_n=a_1+\sum_{k=1}^{n-1}b_k=\sum_{k=1}^{n-1}\left(\frac{1}{3^k}+\frac{1}{2^k}+2\right)=\frac{1}{3}\times\frac{1-\frac{1}{3^{n-1}}}{1-\frac{1}{3}}+\frac{1}{2}\times\frac{1-\frac{1}{2^{n-1}}}{1-\frac{1}{2}}+2(n-1)$

$=\frac{1}{2}-\frac{1}{2\cdot 3^{n-1}}+1-\frac{1}{2^{n-1}}+2n-2=2n-\frac{1}{2\cdot 3^{n-1}}-\frac{1}{2^{n-1}}-\frac{1}{2}$ $(n\geqq 2)$

上式で$n=1$とすると $a_1=2\cdot 1-\frac{1}{2}-\frac{1}{1}-\frac{1}{2}=0$ となるので上式は$n=1$でも成り立つ。

よって$\{a_n\}$の一般式は

$$\mathbf{a_n=2n-\frac{1}{2\cdot 3^{n-1}}-\frac{1}{2^{n-1}}-\frac{1}{2}}\quad (n=1,2,3\dots)$$

4

(1) 左辺 $= \int_{x-1}^{x+2n-1} f(t)\,dt$

$= \int_{x-1}^{x+1} f(t)\,dt + \int_{x+1}^{x+3} f(t)\,dt + \int_{x+3}^{x+5} f(t)\,dt + \cdots + \int_{x+2n-3}^{x+2n-1} f(t)\,dt$

$= x^3 + (x+2)^3 + (x+4)^3 + \cdots + (x+2n-2)^3$

$= \sum_{k=1}^{n} \{x + 2(k-1)\}^3$

= 右辺

証明終

(2) (i) $\int f(x)\,dx = F(x)$ と表す

$\int_{x-1}^{x+1} f(t)\,dt = x^3 \iff F(x+1) - F(x-1) = x^3$

この式の両辺を x で微分すると

$f(x+1) \times (x+1)' - f(x-1) \times (x-1)' = 3x^2$

$f(x+1) - f(x-1) = 3x^2$

証明終

(ii) $f(x) = ax^3 + bx^2 + cx + d$ と表すと.

$f(x+1) - f(x-1) = a(x+1)^3 + b(x+1)^2 + c(x+1) + d - a(x-1)^3 - b(x-1)^2 - c(x-1) - d$

$= 6ax^2 + 2a + 4bx + 2c = 3x^2$

係数比較して. $a = \frac{1}{2}$, $b = 0$. $a + c = 0$ $(a, b, c) = (\frac{1}{2}, 0, -\frac{1}{2})$

$f(x) = \mathbf{\frac{1}{2}x^3 - \frac{1}{2}x}$

(3) (1) で $x = 1$ とすると

$\sum_{k=1}^{n} (1 + 2k - 2)^3 = \sum_{k=1}^{n} (2k-1)^3 = \int_0^{2n} f(t)\,dt$

$= \int_0^{2n} \frac{1}{2}t^3 - \frac{1}{2}t\,dt = \left[\frac{1}{8}t^4 - \frac{1}{4}t^2\right]_0^{2n} = \frac{(2n)^4}{8} - \frac{(2n)^2}{4} = \mathbf{2n^4 - n^2}$