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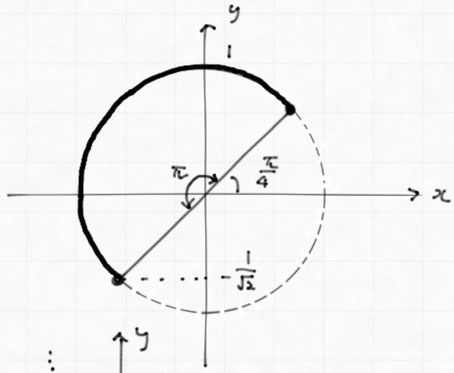
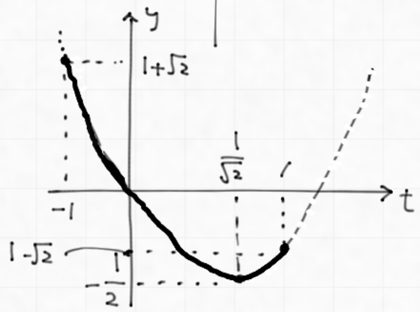
1 (1) $t=\sin x+\cos x=\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$



右図より $-\frac{1}{\sqrt{2}}\leqq\sin\left(x+\frac{\pi}{4}\right)\leqq 1$ ∴ $-1\leqq t\leqq\sqrt{2}$

$t^2=\sin^2 x+2\cos x\sin x+\cos^2 x=1+\sin 2x$ だから

$y=t^2-1-\sqrt{2}t+1=\left(t-\frac{1}{\sqrt{2}}\right)^2-\frac{1}{2}$



右グラフより

$t=-1$, $x=\pi$ のとき yは最大値 $1+\sqrt{2}$ を

$t=\frac{1}{\sqrt{2}}$, のとき yは最小値 $-\frac{1}{2}$ をとる

このとき $\frac{1}{\sqrt{2}}=\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$ より $\sin\left(x+\frac{\pi}{4}\right)=\frac{1}{2}$

$x+\frac{\pi}{4}=\frac{5}{6}\pi$ ∴ $x=\frac{7}{12}\pi$

(2) Cとlの式を連立 $x^2+\left(-\frac{1}{3}x+\frac{5}{3}\right)^2=5$ ⇔ $x^2-x-2=0$ ⇔ $(x-2)(x+1)=0$

$x=-1, 2$ $y=2, 1$ $(x, y)=(-1, 2), (2, 1)$

よって Aは$(-1, 2)$で l_1は $-x+2y=5$

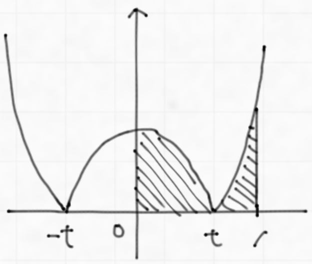
Bは$(2, 1)$で l_2は $2x+y=5$

2式を連立して $(x, y)=(1, 3)$ ∴ $P(1, 3)$

OPの長さは $OP=\sqrt{1^2+3^2}=\sqrt{10}$

$PQ\cdot PR=(OP+\sqrt{5})(OP-\sqrt{5})=OP^2-5=10-5=5$

(3) $S(1)=\int_0^1|x^2-1|dx=\int_0^1-(x^2-1)dx=\left[-\frac{1}{3}x^3+x\right]_0^1=\frac{2}{3}$



$0\leqq t\leqq 1$のとき

$S(t)=\int_0^t|x^2-t^2|dx+\int_t^1|x^2-t^2|dx$

$=\int_0^t-(x^2-t^2)dx+\int_t^1 x^2-t^2dx=\int_t^0 x^2-t^2dx+\int_t^1 x^2-t^2dx$

$=\left[\frac{1}{3}x^3-t^2x\right]_t^0+\left[\frac{1}{3}x^3-t^2x\right]_t^1$

$=\frac{1}{3}-t^2-\left(\frac{1}{3}t^3-t^3\right)\times 2=\frac{4}{3}t^3-t^2+\frac{1}{3}$

$S'(t)=4t^2-2t=2t(2t-1)$ $S'(t)=0$となるのは $t=0, \frac{1}{2}$ 増減は下の通り

t	0	$\cdots$	$\frac{1}{2}$	$\cdots$	1
$S'(t)$		$-$	0	$+$	
$S(t)$		↘		↗	

$S(0)=\frac{1}{3}$, $S(1)=\frac{2}{3}$ $S\left(\frac{1}{2}\right)=\frac{1}{4}$

よって$S(t)$は$t=1$で最大値$\frac{2}{3}$ $t=\frac{1}{2}$で最小値$\frac{1}{4}$をとる

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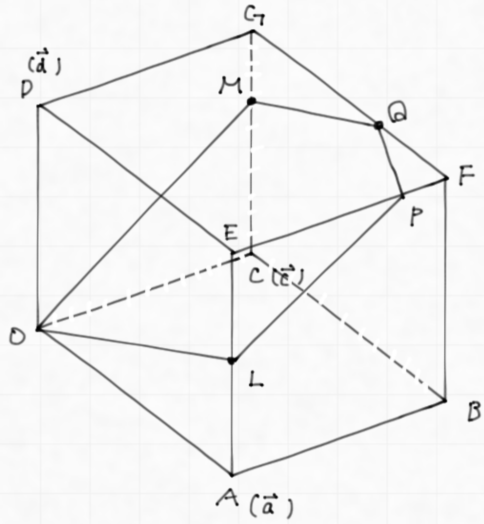
2 $|\vec{a}| = |\vec{c}| = |\vec{d}| = 1 \qquad \vec{a}\cdot\vec{c} = \vec{a}\cdot\vec{d} = \vec{c}\cdot\vec{d} = 0$



(1) $\overrightarrow{OL} = \frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OE} = \frac{1}{2}\vec{a} + \frac{1}{2}(\vec{a}+\vec{d}) = \boldsymbol{\vec{a} + \frac{1}{2}\vec{d}}$

$\overrightarrow{OM} = \frac{1}{4}\overrightarrow{OC} + \frac{3}{4}\overrightarrow{OG} = \frac{1}{4}\vec{c} + \frac{3}{4}(\vec{c}+\vec{d}) = \boldsymbol{\vec{c} + \frac{3}{4}\vec{d}}$

$|\overrightarrow{OL}|^2 = |\vec{a} + \frac{1}{2}\vec{d}|^2 = |\vec{a}|^2 + \vec{a}\cdot\vec{d} + \frac{1}{4}|\vec{d}|^2 = 1 + \frac{1}{4} = \frac{5}{4}$

$|\overrightarrow{OL}| = \boldsymbol{\frac{\sqrt{5}}{2}}$

$|\overrightarrow{OM}|^2 = |\vec{c} + \frac{3}{4}\vec{d}|^2 = 1 + \frac{9}{16} \qquad |\overrightarrow{OM}| = \boldsymbol{\frac{5}{4}}$

$\overrightarrow{OL}\cdot\overrightarrow{OM} = (\vec{a} + \frac{1}{2}\vec{d})\cdot(\vec{c} + \frac{3}{4}\vec{d}) = \frac{3}{8}|\vec{d}|^2 = \boldsymbol{\frac{3}{8}}$

(2) $\cos\angle LOM = \dfrac{\overrightarrow{OL}\cdot\overrightarrow{OM}}{|\overrightarrow{OL}||\overrightarrow{OM}|} = \dfrac{\frac{3}{8}}{\frac{\sqrt{5}}{2}\times\frac{5}{4}} = \dfrac{3}{5\sqrt{5}} = \boldsymbol{\dfrac{3\sqrt{5}}{25}}$

$S_1 = \frac{1}{2}\sqrt{|\overrightarrow{OL}|^2|\overrightarrow{OM}|^2 - (\overrightarrow{OL}\cdot\overrightarrow{OM})^2} = \frac{1}{2}\sqrt{\frac{5}{4}\times\frac{25}{16} - \frac{9}{64}} = \boldsymbol{\frac{\sqrt{29}}{8}}$

(3) $\overrightarrow{OP} = p\overrightarrow{OF} + (1-p)\overrightarrow{OE} = p(\vec{a}+\vec{c}+\vec{d}) + (1-p)(\vec{a}+\vec{d}) = \vec{a} + p\vec{c} + \vec{d} \quad \cdots ①$

$\overrightarrow{OQ} = (1-q)\overrightarrow{OG} + q\overrightarrow{OF} = q\vec{a} + \vec{c} + \vec{d} \quad \cdots ②$

PはOLMの平面上にあるので

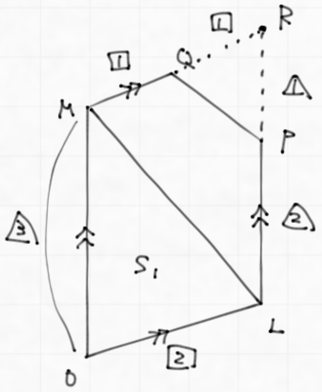
$\overrightarrow{OP} = \alpha\overrightarrow{OL} + \beta\overrightarrow{OM} = \alpha\vec{a} + \frac{1}{2}\alpha\vec{d} + \beta\vec{c} + \frac{3}{4}\beta\vec{d} \quad \cdots ③$

①③より $\vec{a}, \vec{c}, \vec{d}$ は互いに1次独立だから

$1 = \alpha,\ p = \beta,\ 1 = \frac{1}{2}\alpha + \frac{3}{4}\beta \iff \alpha = 1,\ \beta = \frac{2}{3},\ p = \boldsymbol{\frac{2}{3}}$

同様に②③より $q = \alpha,\ 1 = \beta,\ 1 = \frac{1}{2}\alpha + \frac{3}{4}\beta \iff \alpha = \frac{1}{2},\ \beta = 1,\ q = \boldsymbol{\frac{1}{2}}$

$\overrightarrow{OP} = \boldsymbol{\vec{a} + \frac{2}{3}\vec{c} + \vec{d}}, \quad \overrightarrow{OQ} = \boldsymbol{\frac{1}{2}\vec{a} + \vec{c} + \vec{d}}$



(4) $\overrightarrow{LP} = \overrightarrow{OP} - \overrightarrow{OL} = \boldsymbol{\frac{2}{3}\vec{c} + \frac{1}{2}\vec{d}} \qquad \overrightarrow{MQ} = \overrightarrow{OQ} - \overrightarrow{OM} = \boldsymbol{\frac{1}{2}\vec{a} + \frac{1}{4}\vec{d}}$

$|\overrightarrow{LP}|^2 = |\frac{2}{3}\vec{c} + \frac{1}{2}\vec{d}|^2 = \frac{4}{9} + \frac{1}{4} = \frac{25}{36} \qquad |\overrightarrow{LP}| = \frac{5}{6}$

$|\overrightarrow{MQ}|^2 = |\frac{1}{2}\vec{a} + \frac{1}{4}\vec{d}|^2 = \frac{1}{4} + \frac{1}{16} = \frac{5}{16} \qquad |\overrightarrow{MQ}| = \frac{\sqrt{5}}{4}$

$LP : OM = \frac{5}{6} : \frac{5}{4} = 2 : 3, \quad MQ : OL = \frac{\sqrt{5}}{4} : \frac{\sqrt{5}}{2} = 1 : 2$

MQとLPの延長線の交点をRとすると

$OM \parallel LP,\ MQ \parallel OL$ より $MQ : QR = 1 : 1,\ LP : PR = 2 : 1$

$S_2 = \triangle LOM \times 2 - \triangle PQR = 2S_1 - S_1 \times \frac{1}{2} \times \frac{1}{3} = \frac{11}{6}S_1 = \boldsymbol{\frac{11\sqrt{29}}{48}}$