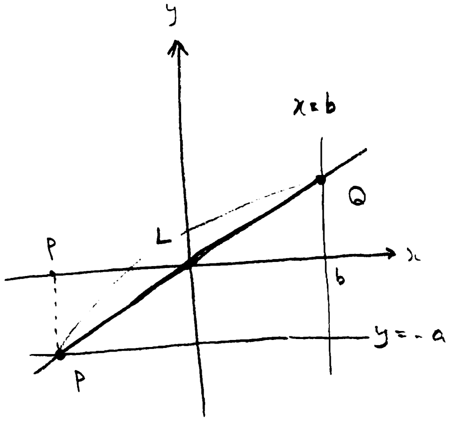
①



(1) 原点を通る直線について $P(p, -a)$ を通るので傾きは $-\frac{a}{p}$

したがってQのy座標は $-\frac{a}{p}b$

よってPQの長さLは三平方の定理より

$$L^2 = (b-p)^2 + \left(-\frac{a}{p}b + a\right)^2$$

を満たす。

$$L^2 = (b-p)^2 + \left(\frac{a}{p}\right)^2(b-p)^2 = (b-p)^2\left(1+\left(\frac{a}{p}\right)^2\right)$$

(2) $L^2 = f(p)$ とおく

$$f'(p) = -2(b-p)\left(1+\frac{a^2}{p^2}\right) + (b-p)^2 \times \left(-2\frac{a^2}{p^3}\right)$$

$$= -2(b-p)\left(1+\frac{a^2}{p^2}+\frac{a^2b}{p^3}-\frac{a^2}{p^2}\right)$$

$$= 2(p-b)\left(1+\frac{a^2b}{p^3}\right)$$

$b>0$, $p<0$ より $p-b<0$. よって $f'(p)=0$ となるのは $1+\frac{a^2b}{p^3}=0$

すなわち $p = -a^{\frac{2}{3}}b^{\frac{1}{3}}$

このとき $f(p)$ の増減は右のようになっている

p	$\cdots$	$-a^{\frac{2}{3}}b^{\frac{1}{3}}$	$\cdots$	0
$f'(p)$	$-$	0	$+$	
$f(p)$	$\searrow$		$\nearrow$	

よって L^2 は $p = -a^{\frac{2}{3}}b^{\frac{1}{3}}$ のとき最小となる

(3) $$f(p_0) = \left(b + a^{\frac{2}{3}}b^{\frac{1}{3}}\right)^2\left(1+\frac{a^2}{a^{\frac{4}{3}}b^{\frac{2}{3}}}\right)$$

$$= \left(b+a^{\frac{2}{3}}b^{\frac{1}{3}}\right)^2\left(1+a^{\frac{2}{3}}b^{-\frac{2}{3}}\right)$$

$$= \left(b^{\frac{1}{3}}\right)^2\left(b^{\frac{2}{3}}+a^{\frac{2}{3}}\right)^2 \times b^{-\frac{2}{3}}\left(b^{\frac{2}{3}}+a^{\frac{2}{3}}\right)$$

$$= \left(c^{\frac{2}{3}}\right)^3 = c^2$$

②

(1) $f'(x) = -e^{-x}\sin x + e^{-x}\cos x$

$g'(x) = -e^{-x}\cos x - e^{-x}\sin x$

よって $\begin{cases} f'(x)+g'(x) = -2e^{-x}\sin x = -2f(x) \\ f'(x)-g'(x) = 2e^{-x}\cos x = 2g(x) \end{cases}$ …①

$I = \int f(x)\,dx = \int -\frac{1}{2}(f'(x)+g'(x))\,dx = -\frac{1}{2}f(x) - \frac{1}{2}g(x) + C_3$

（C_3は積分定数、以下 $C_4, C_5 \ldots$ も同様）

$J = \int g(x)\,dx = \int \frac{1}{2}(f'(x)-g'(x))\,dx = \frac{1}{2}f(x) - \frac{1}{2}g(x) + C_4$

$F(x) = I - J - C_1 = -f(x) + C_3 - C_4 - C_1$

よって $\underline{F(x) = -e^{-x}\sin x}$

$G(x) = J + I - C_2 = -g(x) + C_3 + C_4 - C_2$

よって $\underline{G(x) = -e^{-x}\cos x}$

(2) $I = -\frac{1}{2}e^{-x}(\sin x + \cos x) + C_3$

$J = \frac{1}{2}e^{-x}(\sin x - \cos x) + C_4$

(3) $(k-1)\pi \leqq x \leqq k\pi$ の範囲で $f(x) = g(x)$ となるのは

$e^{-x}\sin x = e^{-x}\cos x$

$\sin x - \cos x = 0$

$\sqrt{2}\sin\left(x - \frac{\pi}{4}\right) = 0$

$\therefore\ x = (k-1)\pi + \frac{\pi}{4}$

よって、もとめる面積は

$$S_k = \left|\int_{(k-1)\pi}^{(k-1)\pi+\frac{\pi}{4}} f(x) - g(x)\,dx\right| + \left|\int_{(k-1)\pi+\frac{\pi}{4}}^{k\pi} f(x) - g(x)\,dx\right|$$

$$= \left|\int_{(k-1)\pi}^{(k-1)\pi+\frac{\pi}{4}} f(x) - g(x)\ dx + \int_{(k-1)\pi+\frac{\pi}{4}}^{k\pi} g(x) - f(x)\ dx\right|$$

$$= \left| \int_{(k-1)\pi}^{(k-1)\pi+\frac{\pi}{4}} -f(x)\,dx + \int_{(k-1)\pi+\frac{\pi}{4}}^{k\pi} f(x)\,dx \right| \qquad (\because ①)$$

$$= \left| \Big[-e^{-x}\sin x\Big]_{(k-1)\pi}^{(k-1)\pi+\frac{\pi}{4}} + \Big[-e^{-x}\sin x\Big]_{k\pi}^{(k-1)\pi+\frac{\pi}{4}} \right|$$

$$= \left| -2e^{-(k-1)\pi-\frac{\pi}{4}}\frac{1}{\sqrt{2}}(-1)^{k-1} + e^{-(k-1)\pi}\times 0 + e^{-k\pi}\times 0 \right|$$

$$= \underline{\sqrt{2}\, e^{-(k-1)\pi-\frac{\pi}{4}}}$$

(4)

S_kは初項 $\sqrt{2}e^{-\frac{\pi}{4}}$，公比 $e^{-\pi}$ の等比数列

$$\sum_{k=1}^{\infty} S_k = \frac{\sqrt{2}e^{-\frac{\pi}{4}}}{1-e^{-\pi}} = \underline{\frac{\sqrt{2}\,e^{\frac{3}{4}\pi}}{e^{\pi}-1}}$$

③ 表を○、裏を×で表す.

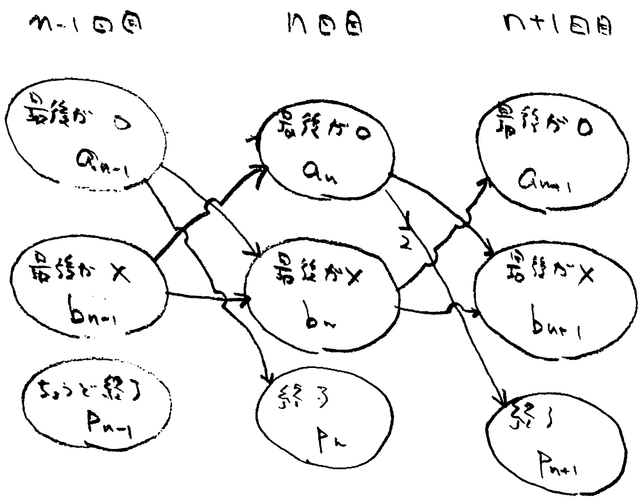
(1) 2回で終わるのは ○○ のときのみ. $P_2 = \left(\frac{1}{2}\right)^2 = \underline{\frac{1}{4}}$

3回 〃 ×○○ 〃 $P_3 = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2 = \underline{\frac{1}{8}}$

4回 〃 ××○○、○×○○ $P_3 = \left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 + \frac{1}{2}\times\frac{1}{2}\times\left(\frac{1}{2}\right)^2 = \frac{1}{8}$

(2) n回目のとき、終了していなくて、n回目が○のときを a_n

×のときを b_n とする.



上の遷移図より

$$a_n = \frac{1}{2}b_{n-1},\quad b_n = \frac{1}{2}a_{n-1} + \frac{1}{2}b_{n-1},\quad p_n = \frac{1}{2}a_{n-1} \qquad (n \geqq 2)$$

などが分かる

$$a_{n-1} = 2p_n \text{ より } \quad b_{n-1} = 2a_n = 4p_{n+1}$$

$$b_n = \frac{1}{2}\times\frac{1}{2}b_{n-2} + \frac{1}{2}b_{n-1}$$

$$4p_{n+2} = \frac{1}{4}\times 4p_n + \frac{1}{2}\times 4p_{n+1}$$

$$\underline{p_{n+1} = \frac{1}{4}p_{n-1} + \frac{1}{2}p_n \qquad (n \geqq 2)}$$

(3) (2)より $p_{n+1} = \frac{1}{4}p_{n-1} + \frac{1}{2}p_n$

$\Leftrightarrow$ $p_{n+1} - \frac{1}{2}p_n = \frac{1}{4}p_{n-1} \geqq 0$

よって　$\frac{P_n}{2} \leqq P_{n+1}$　が成り立つ.

次に　$P_n - P_{n+1} = P_n - \left(\frac{1}{4}P_{n-1} + \frac{1}{2}P_n\right)$

$= \frac{1}{2}\left(P_n - \frac{1}{2}P_{n-1}\right)$

$n \geqq 3$ のとき　$P_n - \frac{1}{2}P_{n-1} \geqq 0$.

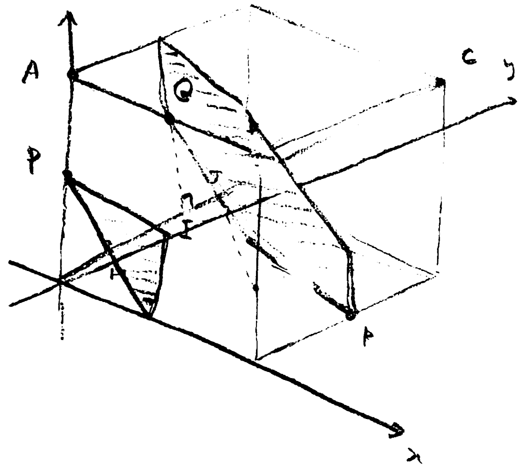
$n = 2$ のとき.　$P_2 - \frac{1}{2}P_1 = \frac{1}{8} - \frac{1}{2} \times \frac{1}{4} = 0$

よって　$n \geqq 2$ において　$P_n - P_{n+1} \geqq 0$

$\therefore\ P_n \geqq P_{n+1}$

以上より.　$\frac{P_n}{2} \leqq P_{n+1} \leqq P_n$　が示された

④



(1)

$$OH = OP\cos\angle POC$$
$$= |\vec{OP}|\frac{\vec{OP}\cdot\vec{OC}}{|\vec{OP}||\vec{OC}|}$$
$$= \frac{p}{\sqrt{3}}$$

$$PH = \sqrt{OP^2 - OH^2} = \sqrt{p^2 - \frac{p^2}{3}} = \underline{p\sqrt{\frac{2}{3}}}$$

(2) $OI = |\vec{OQ}|\cos\angle QOC = \dfrac{\vec{OQ}\cdot\vec{OC}}{|\vec{OC}|} = \dfrac{q+1}{\sqrt{3}}$

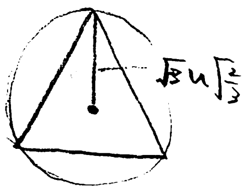
$$QI = \sqrt{OQ^2 - OI^2} = \sqrt{q^2+1-\frac{1}{3}(q+1)^2} = \sqrt{\frac{2}{3}q^2 - \frac{2}{3}q + \frac{2}{3}} = \underline{\sqrt{\frac{2}{3}(q^2-q+1)}}$$

(3) AからCに下ろした垂線の足をH_Aとすると、

(1)より $OH_A = \dfrac{1}{\sqrt{3}}$

同様にBからCに下ろした垂線の足をH_Bとすると (2)より $OH_B = \dfrac{2}{\sqrt{3}}$

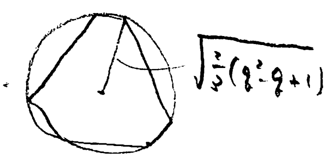
(i) $0 \leqq u \leqq \dfrac{1}{\sqrt{3}}$



$u = \dfrac{p}{\sqrt{3}}$ より $p = \sqrt{3}u$

$$r = p\sqrt{\frac{2}{3}} = \sqrt{2}u$$

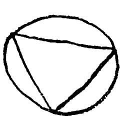
(ii) $\dfrac{1}{\sqrt{3}} \leqq u \leqq \dfrac{2}{\sqrt{3}}$



$\dfrac{q+1}{\sqrt{3}} = u$ より $q = \sqrt{3}u - 1$

$$r = \sqrt{\frac{2}{3}\left((\sqrt{3}u-1)^2 - (\sqrt{3}u-1)+1\right)}$$
$$= \sqrt{2}\sqrt{u^2 - \sqrt{3}u + 1}$$

(iii) $\dfrac{2}{\sqrt{3}} \leqq u \leqq \sqrt{3}$



(i)と同様

$$r = \sqrt{2}\left(\sqrt{3} - u\right)$$

$$r = \begin{cases} \sqrt{2}\,u & \left(0 \leqq u \leqq \frac{1}{\sqrt{3}}\right) \\ \sqrt{2}\sqrt{u^2 - \sqrt{3}u + 1} & \left(\frac{1}{\sqrt{3}} < u \leqq \frac{2}{\sqrt{3}}\right) \\ \sqrt{2}\left(\sqrt{3} - u\right) & \left(\frac{2}{\sqrt{3}} < u \leqq \sqrt{3}\right) \end{cases}$$

(4) 体積をVとすると

$$V = \int_0^{\frac{1}{\sqrt{3}}} \pi\left(\sqrt{2}u\right)^2 du + \int_{\frac{1}{\sqrt{3}}}^{\frac{2}{\sqrt{3}}} \pi\left(\sqrt{2}\sqrt{u^2 - \sqrt{3}u + 1}\right)^2 du + \int_{\frac{2}{\sqrt{3}}}^{\sqrt{3}} \pi\left(\sqrt{2}\left(\sqrt{3} - u\right)\right)^2 du$$

$$= 2\pi\int_0^{\frac{1}{\sqrt{3}}} u^2 du + 2\pi\int_{\frac{1}{\sqrt{3}}}^{\frac{2}{\sqrt{3}}} u^2 - \sqrt{3}u + 1\, du$$

$$= 2\pi\left[\frac{1}{3}u^3\right]_0^{\frac{1}{\sqrt{3}}} + 2\pi\left[\frac{1}{3}u^3 - \frac{\sqrt{3}}{2}u^2 + u\right]_{\frac{1}{\sqrt{3}}}^{\frac{2}{\sqrt{3}}}$$

$$= 2\pi\left(\frac{1}{9\sqrt{3}} - 0 + \frac{8}{9\sqrt{3}} - \frac{2\sqrt{3}}{3} + \frac{2}{\sqrt{3}} - \frac{1}{9\sqrt{3}} + \frac{1}{2\sqrt{3}} - \frac{1}{\sqrt{3}}\right)$$

$$= \frac{\pi}{\sqrt{3}}$$