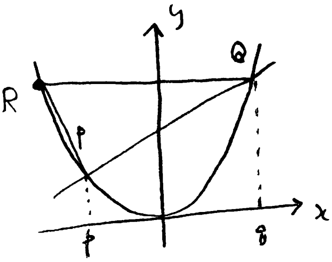
①

(1) $a = \dfrac{q^2 - p^2}{q - p} = \underline{p+q}$



(2) $\tan\theta_1 = 1$ だから $\theta_1 = \underline{\dfrac{\pi}{4}}$

(3) $\angle RPQ = \dfrac{\pi}{3}$ となるので、PRの偏角θ_2は　$\theta_2 = \dfrac{\pi}{4} + \dfrac{\pi}{3} = \underline{\dfrac{7}{12}\pi}$

PRの傾きは　$\tan\theta_2 = \tan\left(\dfrac{\pi}{4} + \dfrac{\pi}{3}\right) = \dfrac{1+\sqrt{3}}{1 - 1\times\sqrt{3}} = \dfrac{4+2\sqrt{3}}{-2} = \underline{-2-\sqrt{3}}$

PRの傾きは(1)より　$p+r$ 、これが $-2-\sqrt{3}$ となったので $p+r=-2-\sqrt{3}$.

QRの傾きは　$\tan\left(\dfrac{\pi}{4} + \dfrac{2}{3}\pi\right) = \dfrac{1-\sqrt{3}}{1 - 1\times(-\sqrt{3})} = \dfrac{-(4-2\sqrt{3})}{2} = \underline{-2+\sqrt{3}} = q+r$

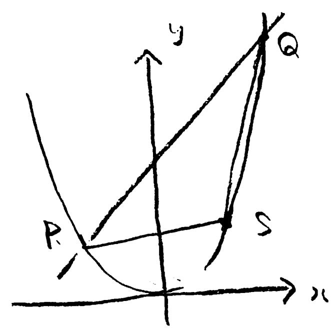
$$\begin{cases} p+q = 1 \\ p+r = -2-\sqrt{3} \\ q+r = -2+\sqrt{3} \end{cases}$$

より　$p = \dfrac{1}{2} - \sqrt{3}$, $q = \dfrac{1}{2} + \sqrt{3}$, $r = -\dfrac{5}{2}$

$\overline{PQ} = \sqrt{2}(q-p) = 2\sqrt{6}$.

$\triangle PQR = \dfrac{1}{2} \times (2\sqrt{6})^2 \times \sin\dfrac{\pi}{3} = \dfrac{24}{2} \times \dfrac{\sqrt{3}}{2} = \underline{6\sqrt{3}}$

(4) PQの傾きが2より　$p+q=2$ …①



$\overrightarrow{SQ} = (q-1,\ q^2-1)$

$\overrightarrow{SP} = (p-1,\ p^2-1)$

$\angle PQS = \dfrac{\pi}{2}$ なので $\overrightarrow{SP}\cdot\overrightarrow{SQ} = 0$ となる.

$(q-1)(p-1) + (q^2-1)(p^2-1) = 0$

$(q-1)(p-1)\{1 + (p+1)(q+1)\} = 0$.

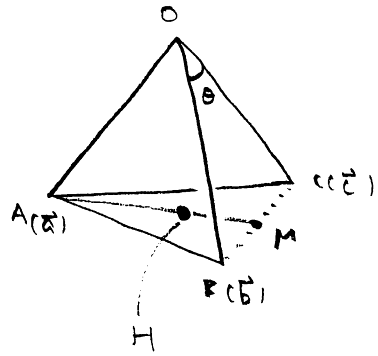
$p \neq 1,\ q \neq 1$ だから　$(p+1)(q+1)+1 = 0$ …②

①, ②より　$(p, q) = (1-\sqrt{5},\ 1+\sqrt{5})$

$PS = \sqrt{5 + (5-2\sqrt{5})^2} = \sqrt{50-20\sqrt{5}}$, $QS = \sqrt{50+20\sqrt{5}}$

$\triangle PQS = \dfrac{1}{2}\cdot PS\cdot QS = \dfrac{1}{2}\sqrt{50^2 - (20\sqrt{5})^2} = \underline{5\sqrt{5}}$

2



$|\vec{a}| = 1,\ |\vec{b}| = 1,\ |\vec{c}| = 1$

$\vec{a}\cdot\vec{b} = 1\times 1\times\cos 60^\circ = \frac{1}{2},\ \vec{a}\cdot\vec{c} = \frac{1}{2}$

$\vec{b}\cdot\vec{c} = 1\times 1\times\cos\theta = \cos\theta$

(1) MはBCの中点 $\overrightarrow{OM} = \underline{\frac{1}{2}\vec{b} + \frac{1}{2}\vec{c}}$

$\overrightarrow{AM} = \overrightarrow{OM} - \overrightarrow{OA} = \underline{-\vec{a} + \frac{1}{2}\vec{b} + \frac{1}{2}\vec{c}}$

(2) $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos 60^\circ = \underline{\frac{1}{2}}$　$\vec{a}\cdot\vec{c} = |\vec{a}||\vec{c}|\cos 60^\circ = \underline{\frac{1}{2}}$

$|\vec{b}+\vec{c}|^2 = |\vec{b}|^2 + 2\vec{b}\cdot\vec{c} + |\vec{c}|^2 = 1 + 2\times|\vec{b}||\vec{c}|\cos\theta + 1 = \underline{2+2\cos\theta}$

(3) $\overrightarrow{OH}\cdot\overrightarrow{BC} = \left\{(1-t)\vec{a} + t\left(\frac{1}{2}\vec{b} + \frac{1}{2}\vec{c}\right)\right\}\cdot(\vec{c}-\vec{b})$

$= (1-t)\vec{a}\cdot\vec{c} + \frac{1}{2}t\vec{b}\cdot\vec{c} + \frac{1}{2}t|\vec{c}|^2 - (1-t)\vec{a}\cdot\vec{b} - \frac{1}{2}t|\vec{b}|^2 - \frac{1}{2}t\vec{b}\cdot\vec{c}$

$= \frac{1}{2}(1-t) + \frac{1}{2}t\cos\theta + \frac{1}{2}t - \frac{1}{2}(1-t) - \frac{1}{2}t - \frac{1}{2}t\cos\theta$

$= 0.$

$\therefore$ OH $\perp$ BC である.

$\overrightarrow{OH}\perp\overrightarrow{AM}$ が成り立つとき $\overrightarrow{OH}\cdot\overrightarrow{AM} = 0$ だから

$\left\{(1-t)\vec{a} + \frac{1}{2}t\vec{b} + \frac{1}{2}t\vec{c}\right\}\cdot\left\{-\vec{a} + \frac{1}{2}\vec{b} + \frac{1}{2}\vec{c}\right\} = 0$

$(1-t)|\vec{a}|^2 + \frac{1}{2}(1-t)\vec{a}\cdot\vec{b} + \frac{1}{2}(1-t)\vec{a}\cdot\vec{c} - \frac{1}{2}t\vec{a}\cdot\vec{b} + \frac{1}{4}t|\vec{b}|^2 + \frac{1}{4}t\vec{b}\cdot\vec{c}$

$- \frac{1}{2}t\vec{a}\cdot\vec{c} + \frac{1}{4}t\vec{b}\cdot\vec{c} + \frac{1}{4}t|\vec{c}|^2 = 0$

$(1+\cos\theta)t = 1$

$\underline{t = \frac{1}{1+\cos\theta}}$

(4) $h = |\overrightarrow{OH}| = \left|\frac{\cos\theta}{1+\cos\theta}\vec{a} + \frac{1}{2(1+\cos\theta)}\vec{b} + \frac{1}{2(1+\cos\theta)}\vec{c}\right|$

$= \frac{1}{2(1+\cos\theta)}\left|2\cos\theta\vec{a} + \vec{b} + \vec{c}\right|$

$|2\cos\theta\vec{a} + \vec{b} + \vec{c}|^2 = 4\cos^2\theta + 1 + 1 + 4\cos\theta + 2\cos\theta = 4\cos^2\theta + 6\cos\theta + 2$

よって $h = \dfrac{\sqrt{2}\sqrt{(\cos\theta+1)(2\cos\theta+1)}}{2(1+\cos\theta)} = \underline{\sqrt{\dfrac{2\cos\theta+1}{2(1+\cos\theta)}}}$

$\overrightarrow{OM} \perp \overrightarrow{AM}$ が成り立つとき

$$\overrightarrow{OM}\cdot\overrightarrow{AM} = \frac{1}{2}(\vec{b}+\vec{c})\cdot\left(-\vec{a}+\frac{1}{2}\vec{b}+\frac{1}{2}\vec{c}\right)$$

$$= \frac{1}{4}(\vec{b}+\vec{c})\cdot(\vec{b}+\vec{c}-2\vec{a})$$

$$= \frac{1}{4}\left(1+\cos\theta-1+\cos\theta+1-1\right) = \frac{1}{2}\cos\theta = 0$$

$\therefore\ \theta = \underline{90^\circ}$

このとき $h = \sqrt{\dfrac{2\times 0+1}{2(1+0)}} = \underline{\dfrac{1}{\sqrt{2}}}$

③ (1) $a_{n+1} = (\sqrt{2}+1)a_n + 1$

$a_{n+1} + \frac{1}{\sqrt{2}} = (\sqrt{2}+1)\left(a_n + \frac{1}{\sqrt{2}}\right)$

$\left\{a_n + \frac{1}{\sqrt{2}}\right\}$は初項 $a_1 + \frac{1}{\sqrt{2}}$，公比 $(\sqrt{2}+1)$の等比数列なので，その一般項は

$$a_n + \frac{1}{\sqrt{2}} = \left(1+\frac{1}{\sqrt{2}}\right)(\sqrt{2}+1)^{n-1} = \frac{\sqrt{2}}{2}(\sqrt{2}+1)^n$$

$$\underline{a_n = \frac{\sqrt{2}}{2}(\sqrt{2}+1)^n - \frac{1}{\sqrt{2}}}$$

$\underline{\lim_{n\to\infty} a_n = \infty}$ $(\because \sqrt{2}+1>1)$

(2)

$$\lim_{n\to\infty}\left(\frac{1}{n^2+1^2} + \frac{2}{n^2+2^2} + \cdots + \frac{n}{n^2+n^2}\right)$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n^2+k^2}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{\frac{k}{n}}{1+\left(\frac{k}{n}\right)^2}$$

$$= \int_0^1 \frac{x}{1+x^2}dx = \frac{1}{2}\int_0^1\frac{(1+x^2)'}{(1+x^2)}dx = \frac{1}{2}\left[\log(1+x^2)\right]_0^1$$

$$= \underline{\frac{1}{2}\log 2}$$

(3)

$$\int_0^1 \pi y^2 dx = \int_0^1 \pi(1-\sqrt{x})^4 dx$$

$$= \pi\int_0^1 1 - 4\sqrt{x} + 6x - 4x\sqrt{x} + x^2 dx$$

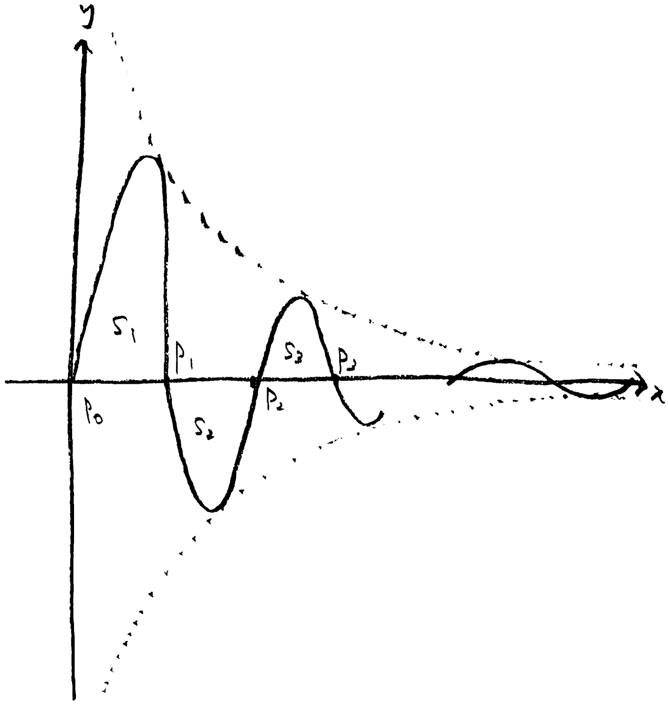
$$= \pi\left[x - \frac{8}{3}x^{\frac{3}{2}} + 3x^2 - \frac{8}{5}x^{\frac{5}{2}} + \frac{1}{3}x^3\right]_0^1$$

$$= \pi\left(1 - \frac{8}{3} + 3 - \frac{8}{5} + \frac{1}{3}\right) = \underline{\frac{1}{15}\pi}$$

⑥

(1) $e^{-x} > 0$ だから $f(x) = 0$ となるのは $\sin x = 0$ のときのみ。

よって $\underline{x_n = n\pi}$ //



(2)

$$S_n = \int_{x_{n-1}}^{x_n} |f(x)|\,dx$$

$$= \int_{(n-1)\pi}^{n\pi} |e^{-x}\sin x|\,dx$$

$$= \left| \int_{(n-1)\pi}^{n\pi} e^{-x}\sin x\,dx \right|$$

ここで $\int e^{-x}\sin x\,dx = -e^{-x}\sin x + \int e^{-x}\cos x\,dx$

$= -e^{-x}\sin x - e^{-x}\cos x - \int e^{-x}\sin x\,dx$

より $\int e^{-x}\sin x\,dx = -\frac{1}{2}e^{-x}(\sin x + \cos x) + C$ （Cは積分定数）

$$S_n = \left| \frac{1}{2}\left[e^{-x}(\sin x + \cos x) \right]_{(n-1)\pi}^{n\pi} \right|$$

$$= \left| \frac{1}{2}e^{-n\pi}\cos n\pi - \frac{1}{2}e^{-(n-1)\pi}\cos (n-1)\pi \right|$$

$$= \left| \frac{1}{2}e^{-n\pi}(-1)^n - \frac{1}{2}e^{-(n-1)\pi}(-1)^{n-1} \right|$$

$$= \frac{1}{2}e^{-n\pi} + \frac{1}{2}e^{-(n-1)\pi} = \underline{\frac{1}{2}e^{-n\pi}(1+e^{\pi})}$$ //

(3) $I_n = \sum_{k=1}^{n} S_k = \frac{1}{2}(1+e^{\pi})\sum_{k=1}^{n} e^{-k\pi} = \frac{1}{2}(1+e^{\pi}) \times \frac{e^{-\pi}(1-e^{-n\pi})}{1-e^{-\pi}} = \frac{(1+e^{\pi})(1-e^{-n\pi})}{2(e^{\pi}-1)}$

$$\lim_{n\to\infty} I_n = \underline{\frac{1+e^{\pi}}{2(e^{\pi}-1)}}$$ // $(\because e^{-n\pi} \to 0)$