① (1)

赤がでるのは $\frac{1}{10}$ 　白は $\frac{9}{10}$

$${}_{10}C_8 \times \left(\frac{1}{10}\right)^8\left(\frac{9}{10}\right)^2 = \frac{10 \cdot 9}{2} \times \left(\frac{1}{10}\right)^8 \times \left(\frac{9}{10}\right)^2 = \frac{5 \times 9^3}{10^{10}} = \frac{9^3}{2 \cdot 10^9}$$

(2) 赤がでるのは $\frac{1}{n}$, 白は $\frac{n-1}{n}$

$${}_nC_8 \cdot \left(\frac{1}{n}\right)^8\left(\frac{n-1}{n}\right)^{n-8} = \frac{n!}{(n-8)!\,8!} \times \frac{(n-1)^{n-8}}{n^n}$$

(3)

$$\lim_{n\to\infty} \frac{n!}{(n-8)!\,8!} \times \frac{(n-1)^{n-8}}{n^n}$$

$$= \lim_{n\to\infty} \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)}{8! \times n^8}\left(\frac{1}{\frac{n}{n-1}}\right)^{n-8}$$

$$= \lim_{n\to\infty} \frac{(1-\frac{1}{n})(1-\frac{2}{n})(1-\frac{3}{n})(1-\frac{4}{n})(1-\frac{5}{n})(1-\frac{6}{n})(1-\frac{7}{n})}{8!}\left(\frac{1}{1+\frac{1}{n-1}}\right)^{(n-1)\times\frac{1-\frac{8}{n}}{1-\frac{1}{n}}}$$

$$= \frac{1}{8!} \times \left(\frac{1}{e}\right)^1 = \underline{\frac{1}{40320e}}$$

②

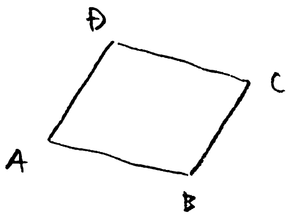
(1) (i) $\vec{a}\cdot\vec{d} = \vec{a}\cdot\vec{b} - \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}|\vec{a}|^2 = 0$

(ii) $\vec{e}\cdot\vec{d} = \left(\vec{c} - \dfrac{\vec{a}\cdot\vec{c}}{|\vec{c}|^2}\vec{a} - \dfrac{\vec{d}\cdot\vec{c}}{|\vec{d}|^2}\vec{d}\right)\cdot\vec{d}$

$= \vec{c}\cdot\vec{d} - \dfrac{(\vec{a}\cdot\vec{c})\times(\vec{a}\cdot\vec{d})}{|\vec{c}|^2} - \vec{c}\cdot\vec{d}$

$= 0 \qquad (\because\ \vec{a}\cdot\vec{d} = 0)$

(2)



Dを表す複素数をδとして

$\delta - \alpha = \gamma - \beta$

$\delta = \alpha + \gamma - \beta$

$= (1+i) + (-1-2i) - (3-7i)$

$= \underline{-3+6i}$

(3)

(i) a_kと表すと $a_{k+1} - a_k = 10^k$

$\therefore\ a_k = a_1 + \sum_{l=1}^{k-1} 10^l \qquad (k \geqq 2)$

$= 1 + \dfrac{10(10^{k-1}-1)}{10-1}$

$= 1 + \dfrac{10}{9}(10^{k-1}-1)$

$= \dfrac{1}{9}\times 10^k - \dfrac{1}{9}$

ここで $k=1$ とすると $\dfrac{1}{9}\times 10 - \dfrac{1}{9} = 1$ となり、これは $k=1$ でも成り立つ

$\therefore$ 第k項は $\underline{\dfrac{1}{9}(10^k-1)}$

(ii) $\sum_{k=1}^{n} \dfrac{1}{9}(10^k-1) = -\dfrac{1}{9}n + \dfrac{10}{9}\times\dfrac{10^n-1}{10-1} = \underline{\dfrac{10^{n+1}-9n-10}{81}}$

③ (1) $2016 = 2^5 \times 3^2 \times 7^1$

(i) 約数は $2^a \cdot 3^b \cdot 7^c$ と表すことができ、

$0 \leqq a \leqq 5,\ 0 \leqq b \leqq 2,\ 0 \leqq c \leqq 1$ より、

$(5+1)(2+1)(1+1) = 6 \times 3 \times 2 = 36$

$\therefore$ 36通り。

(ii)

$2^0 \cdot 3^0 \cdot 7^0 + 2^0 \cdot 3^0 \cdot 7^1 + \cdots + 2^5 \cdot 3^2 \cdot 7^1$

$= 2^0 \cdot 3^0 \cdot (7^0 + 7^1) + 2^0 \cdot 3^1 \cdot (7^0 + 7^1) + \cdots + 2^5 \cdot 3^2 (7^0 + 7^1)$

$= 2^0 \cdot (3^0 + 3^1 + 3^2)(7^0 + 7^1) + \cdots + 2^5 \cdot (3^0 + 3^1 + 3^2)(7^0 + 7^1)$

$= (2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5)(3^0 + 3^1 + 3^2)(7^0 + 7^1)$

$= \dfrac{2^6 - 1}{2-1} \times \dfrac{3^3 - 1}{3-1} \times 8 = 63 \times \dfrac{26}{\cancel{2}} \times \cancel{8}^{4} = 6552$

(2)

$\log_3 \left(a + \dfrac{3}{b}\right) + \log_3 \left(b + \dfrac{3}{a}\right)$

$= \log_3 \left(a + \dfrac{3}{b}\right)\left(b + \dfrac{3}{a}\right)$

$= \log_3 \left(ab + \dfrac{9}{ab} + 6\right)$

$\geqq \log_3 \left(2\sqrt{ab \times \dfrac{9}{ab}} + 6\right)$

$= \log_3 12$

$= \underline{1 + 2\log_3 2}$ （等号は $ab = \dfrac{9}{ab} \Leftrightarrow a^2b^2 = 9$ のとき）

(3)

$(\sin\theta + \cos\theta)^2 = \left(\dfrac{2}{3}\right)^2$ より $\sin\theta\cos\theta = -\dfrac{5}{18}$

$(\cos\theta - \sin\theta)^2 = 1 - 2\cos\theta\sin\theta = 1 + \dfrac{5}{9} = \dfrac{14}{9}$

$-\dfrac{\pi}{2} < \theta < 0$ のとき $\cos\theta > 0,\ \sin\theta < 0$ なので $\cos\theta - \sin\theta > 0$.

よって. $\cos\theta - \sin\theta = \dfrac{\sqrt{14}}{3}$.

$$\cos^3\theta - \sin^3\theta = (\cos\theta - \sin\theta)(\cos^2\theta + \cos\theta\sin\theta + \sin^2\theta)$$

$$= \frac{\sqrt{14}}{3}\left(1 - \frac{5}{18}\right) = \frac{13\sqrt{14}}{54}$$

(4)

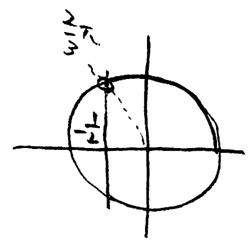
$$\sin 3x + \sin 2x + \sin x = 2\sin\frac{3x+x}{2}\cos\frac{3x-x}{2} + \sin 2x$$

$$= \sin 2x\,(2\cos x + 1)$$

$0 \leqq x \leqq \pi$ において. $\sin 2x$ が正となるのは $0 < x < \frac{\pi}{2}$

$2\cos x + 1$ が正となるのは $0 \leqq x \leqq \frac{2}{3}\pi$.

よって与式は



$$\int_0^{\pi} \left|\sin 3x + \sin 2x + \sin x\right| dx$$

$$= \int_0^{\frac{\pi}{2}} \sin 3x + \sin 2x + \sin x\, dx + \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} -\sin 3x - \sin 2x - \sin x\, dx$$

$$+ \int_{\frac{2}{3}\pi}^{\pi} \sin 3x + \sin 2x + \sin x\, dx$$

$$= \left[\frac{1}{3}\cos 3x + \frac{1}{2}\cos 2x + \cos x\right]_{\frac{\pi}{2}}^{0} + \left[\frac{1}{3}\cos 3x + \frac{1}{2}\cos 2x + \cos x\right]_{\frac{\pi}{2}}^{\frac{2\pi}{3}}$$

$$+ \left[\frac{1}{3}\cos 3x + \frac{1}{2}\cos 2x + \cos x\right]_{\pi}^{\frac{2}{3}\pi}$$

$$= \left(\frac{1}{3} + \frac{1}{2} + 1\right) - \left(0 - \frac{1}{2} + 0\right) \times 2 + \left(\frac{1}{3} - \frac{1}{4} - \frac{1}{2}\right) \times 2 - \left(-\frac{1}{3} + \frac{1}{2} - 1\right)$$

$$= \frac{11}{6} + 1 - \frac{5}{6} + \frac{5}{6} = \frac{17}{6}$$

④

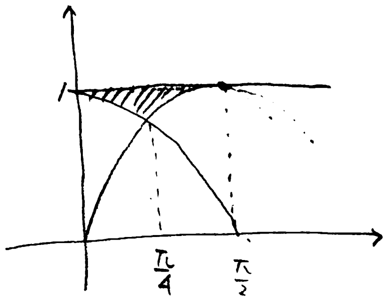
(1)

(i) $S = 2\int_0^{\frac{\pi}{4}} 1-\cos x \, dx$

$= \frac{\pi}{2} - 2[\sin x]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{2} - 2\left(\frac{\sqrt{2}}{2} - 0\right)$

$= \underline{\frac{\pi}{2} - \sqrt{2}}$ //



(ii)

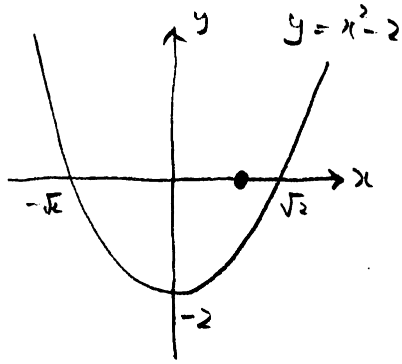
$V = 2\int_0^{\frac{\pi}{4}} \pi(1-\cos x)^2 \, dx$

$= 2\pi \int_0^{\frac{\pi}{4}} 1 - 2\cos x + \frac{1+\cos 2x}{2} \, dx$

$= 2\pi \left[\frac{3}{2}x - 2\sin x + \frac{1}{4}\sin 2x \right]_0^{\frac{\pi}{4}}$

$= 2\pi \left(\frac{3}{8}\pi - \sqrt{2} + \frac{1}{4} - 0 - 0 - 0 \right)$

$= \underline{\frac{\pi(3\pi - 8\sqrt{2} + 2)}{4}}$ //



(2) Pを(t, t^2-2)とおくと

$(1, 0)$とPとのきょり(これを$l(t)$とする)は、

$l(t) = \sqrt{(t-1)^2 + (t^2-2)^2} = \sqrt{t^4 - 3t^2 - 2t + 5}$

$\frac{1 \pm \sqrt{1+2}}{2}$

$f(t) = l(t)^2 = t^4 - 3t^2 - 2t + 5$ とおく。

$f'(t) = 2(2t^3 - 3t - 1) = 2(t+1)(2t^2 - 2t - 1) = (t+1)(2t - 1 + \sqrt{3})(2t - 1 - \sqrt{3})$

$f(t)$の増減は下のようになる

t	$\cdots$	-1	$\cdots$	$\frac{1-\sqrt{3}}{2}$	$\cdots$	$\frac{1+\sqrt{3}}{2}$	$\cdots$
$f'(t)$	$-$	0	$+$	0	$-$	0	$+$
$f(t)$	↘		↗		↘		↗

$f(-1) = 1 - 3 + 2 + 5 = 5.$

ここで $f(t)$ は

$$2f(t) = (2t^2-2t-1)\left(t^2+t-\frac{3}{2}\right) - 6t + \frac{23}{2}$$

$$
\begin{array}{rrr|rrrrr}
 & & & & & 1 & 1 & -\frac{3}{2} \\
\hline
2 & -2 & -1 & 2 & 0 & -6 & -4 & 10 \\
 & & & 2 & -2 & -1 & & \\
\hline
 & & & & 2 & -5 & -4 & \\
 & & & & 2 & -2 & -1 & \\
\hline
 & & & & & -3 & -3 & 10 \\
 & & & & & -3 & +3 & +\frac{3}{2} \\
\hline
 & & & & & & -6 & +\frac{23}{2}
\end{array}
$$

となるので

$$2f\left(\frac{1+\sqrt{3}}{2}\right) = -6\times\frac{1+\sqrt{3}}{2} + \frac{23}{2}$$

$$= \frac{17}{2} - 3\sqrt{3}$$

$$f\left(\frac{1+\sqrt{3}}{2}\right) = \frac{17}{4} - \frac{3}{2}\sqrt{3} < 5, \quad y = \left(\frac{1+\sqrt{3}}{2}\right)^2 - 2 = \frac{-2+\sqrt{3}}{2}$$

∴ 最小となるのは $P\left(\frac{1+\sqrt{3}}{2}, \frac{-2+\sqrt{3}}{2}\right)$