$\boxed{1}$

(1) $a = 100\log_{10} 3\cdot 5\cdot 7 = 100\left(\log_{10}3 + \log_{10}\frac{10}{2} + \log_{10}7\right)$

$= 100\left(0.4771 + 1 - 0.3010 + 0.8451\right) = 202.12$ $\qquad \underline{202}$

(2) $\log_{10}3^{500} = 500\log_{10}3 = 500\times 0.4771 = 238.55$

$3^{500} = 10^{238.55} = 10^{238}\times 10^{0.55}$

$\log_{10}3 = 0.4771,\quad \log_{10}4 = 0.6020$ だから

$3 < 10^{0.55} < 4$ $\qquad$ よって 3^{500}は $\underline{239}$ 桁の数で

最高位の数は $\underline{3}$

(3) $3^1 \equiv 3 \pmod{10}$ 以下全て mod 10

$3^2 \equiv 9$

$3^3 \equiv 7$

$3^4 \equiv 1$ $\qquad\qquad \underline{4\text{種}}$

$3^5 \equiv 3^4\cdot 3 \equiv 3$

$\vdots$

$3^{500} \equiv (3^4)^{125} \equiv \underline{1}$

(4) $3^5 = 243 \equiv 3 \pmod{60}$ 以下同じ

$3^{500} \equiv (3^5)^{100} \equiv 3^{100} \equiv (3^5)^{20} \equiv 3^{20} \equiv (3^5)^4 \equiv 3^4 \equiv 81 \equiv \underline{21}$

(5) $172 \div 53 = 3 \cdots 13$ $\qquad 172 = 3\times 53 + 13$

$53 \div 13 = 4 \cdots 1$ $\qquad 53 = 4\times 13 + 1$

$53 = 4\times(172 - 3\times 53) + 1$

$4\times 172 - 13\times 53 + 1 = 0$

$172\times(-4) - 53\times(-13) = 1$

$$\begin{array}{rl} & 172x - 53y = 1 \\ -) & 172(-4) - 53(-13) = 1 \\ \hline & 172(x+4) - 53(y+13) = 0 \end{array}$$

$$\begin{cases} x+4 = 53n \\ y+13 = 172n \end{cases}$$

$x = 53n - 4,\quad y = 172n - 13$

$n=0$ のとき 絶対値は最小. $\underline{x=-4,\quad y=-13}$

$\boxed{2}$

$\alpha = 3(\cos\theta + i\sin\theta)$

$\beta = 3\left\{\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\right) + i\sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)\right\}$

$= 3\sqrt{2}\left(\cos\left(\theta + \frac{\pi}{4}\right) + i\sin\left(\theta + \frac{\pi}{4}\right)\right)$

(1) $\left|\frac{\beta}{\alpha}\right| = \frac{|\beta|}{|\alpha|} = \underline{\sqrt{2}}$ //

$\arg\frac{\beta}{\alpha} = \arg\beta - \arg\alpha = \theta + \frac{\pi}{4} - \theta = \underline{\frac{1}{4}}\pi$ //

$\beta - \alpha = 3(-\sin\theta + i\cos\theta) = 3(\sin(-\theta) + i\cos(-\theta))$

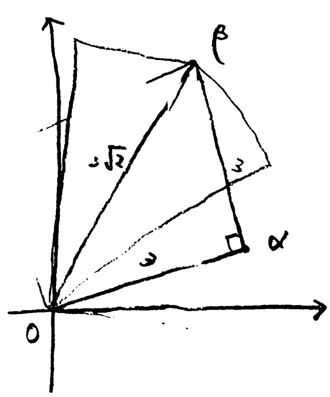
$= 3\left(\cos\left(\frac{\pi}{2}+\theta\right) + i\sin\left(\frac{\pi}{2}+\theta\right)\right)$

なので

$|\beta - \alpha| = \underline{3}$ //

$\arg\frac{\beta-\alpha}{-\alpha} = \arg(\beta-\alpha) - \arg(-\alpha) = \frac{\pi}{2}+\theta - (\pi+\theta) = -\underline{\frac{1}{2}}\pi$ //

(2) 右図より 直径 $\underline{3\sqrt{2}}$ //



(3) $|\gamma| = \frac{9\sqrt{3}}{|\beta|} = \frac{3\sqrt{3}}{\sqrt{2}} = \frac{3\sqrt{6}}{2} = 3\sqrt{2} \times \frac{\sqrt{3}}{2}$

$\arg\gamma = \arg\frac{-9\sqrt{3}}{\beta} = \pi - \arg\beta$

$= \pi - \theta - \frac{\pi}{4} = \frac{3}{4}\pi - \theta \quad \cdots ①$

γが円周上にあるとき$\angle OCB = \frac{\pi}{2}$となるので

$\arg\frac{\beta-\gamma}{-\gamma} = \pm\underline{\frac{1}{2}}\pi$ //

であり、これより

$|\gamma| : |\beta| = \frac{3\sqrt{6}}{2} : 3\sqrt{2} = \sqrt{3} : 2$

であり$\angle OCB = \frac{\pi}{2}$だから、$\triangle OBC$は $OB : OC : BC = 2 : \sqrt{3} : 1$

の直角三角形

よって $\arg\gamma = \arg\beta \pm \frac{\pi}{6} = \theta + \frac{\pi}{4} \pm \frac{\pi}{6}$

①とあわせて. $\frac{3}{4}\pi - \theta = \theta + \frac{\pi}{4} \pm \frac{\pi}{6}$

$$\theta = \frac{1}{6}\pi,\ \frac{1}{3}\pi$$

$$\underline{\theta_1 = \frac{1}{6}\pi}_{\,/\!/},\ \underline{\theta_2 = \frac{1}{3}\pi}_{\,/\!/}$$

$\theta = \theta_1$ のとき

$$\alpha = 3\left(\cos\frac{1}{6}\pi + i\sin\frac{1}{6}\pi\right) = \frac{3}{2}\left(\sqrt{3} + i\right)$$

$$\beta = 3\left\{\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) + i\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right)\right\}$$

$$= \frac{3}{2}\left\{(\sqrt{3}-1) + (\sqrt{3}+1)i\right\}$$

$$\gamma = \frac{3\sqrt{6}}{2}\left\{\cos\left(\frac{3}{4}\pi - \frac{1}{6}\pi\right) + i\sin\left(\frac{3}{4}\pi - \frac{1}{6}\pi\right)\right\}$$

$$= \frac{3\sqrt{6}}{2}\left(-\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\times\frac{1}{2} + i\left(\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\times\frac{1}{2}\right)\right)$$

$$= \frac{3\cancel{\sqrt{6}}^{\sqrt{3}}}{4\cancel{\sqrt{2}}}\left(-\sqrt{3}+1 + i(\sqrt{3}+1)\right)$$

$$= \frac{3}{4}\left(\sqrt{3}-3 + (\sqrt{3}+3)i\right)$$

$\boxed{3}$

(1) $f_0(x) = \sin x$, $f_1(x) = \sin 2x$

$\sin x = \sin 2x \iff \sin x(1 - 2\cos x) = 0$

$x = \ , \pi, \cdots \frac{\pi}{3}, \frac{5}{3}\pi \cdots$

このうちで最小のものは $\frac{\pi}{3}$

$\therefore\ x_0 = \underline{\frac{\pi}{3}}$

$f_n(x) = f_{n+1}(x) \iff \sin(2^n x) = \sin(2^{n+1}x)$

$\iff \sin(2^n x) - 2\sin(2^n x)\cos 2^n x = 0$

$\iff \sin(2^n x)(1 - 2\cos 2^n x) = 0.$

$2^n x = m\pi$, $2^n x = \frac{\pi}{3} + m\pi, -\frac{\pi}{3} + m\pi.$

最小のものは $x = \frac{\pi}{3 \cdot 2^n}$ $\therefore\ x_1 = \underline{\frac{\pi}{6}}$, $x_2 = \underline{\frac{\pi}{12}}$

(2)

$$S_n = \int_{\frac{\pi}{3\cdot 2^{n+1}}}^{\frac{\pi}{3\cdot 2^n}} \sin(2^{n+1}x) - \sin(2^n x)\,dx$$

$$= \left[-\frac{1}{2^{n+1}}\cos(2^{n+1}x) + \frac{1}{2^n}\cos(2^n x)\right]_{\frac{\pi}{3\cdot 2^{n+1}}}^{\frac{\pi}{3\cdot 2^n}}$$

$$= -\frac{1}{2^{n+1}}\cos\frac{2}{3}\pi + \frac{1}{2^n}\cos\frac{\pi}{3} + \frac{1}{2^{n+1}}\cos\frac{\pi}{3} - \frac{1}{2^n}\cos\frac{\pi}{6}$$

$$= \frac{1}{2^{n+2}} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} - \frac{\sqrt{3}}{2^{n+1}} = \frac{4 - 2\sqrt{3}}{2^{n+2}} = \frac{2-\sqrt{3}}{2^{n+1}}$$

$S_0 = \frac{2-\sqrt{3}}{2^1} = \underline{\frac{2-\sqrt{3}}{2}}$, $S_1 = \underline{\frac{2-\sqrt{3}}{4}}$

$S_n = \frac{2-\sqrt{3}}{2^{n+1}} = \frac{1}{2^{n+1}(2+\sqrt{3})}$

$n = 4$ $S_4 = \frac{1}{2^5(2+\sqrt{3})} < \frac{1}{32 \times 3.8} <$

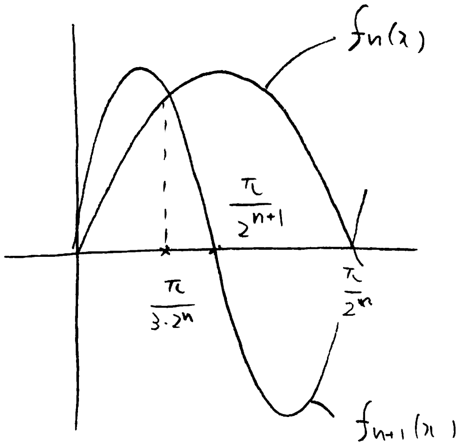
$S_3 = \frac{1}{2^4(2+\sqrt{3})} > \frac{1}{16 \times 3.7} > \frac{1}{100}$ $n = \underline{4}$

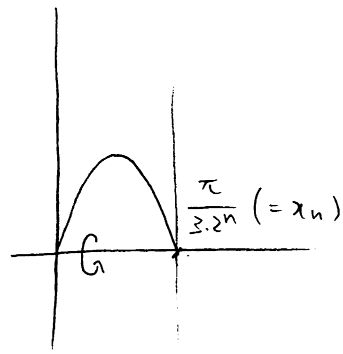
(4) $\displaystyle\int_0^{\frac{\pi}{3}} \sin^2 x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{3}} 1-\cos 2x\,dx = \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{3}}$

$\displaystyle = \frac{1}{2}\left(\frac{\pi}{3} - \frac{1}{2}\times\frac{\sqrt{3}}{2}\right) = \underline{\frac{\pi}{6} - \frac{\sqrt{3}}{8}}$

(5) $\displaystyle\int_0^{\frac{\pi}{3}} \sin x \sin 2x\,dx = 2\int_0^{\frac{\pi}{3}} \sin^2 x\cos x\,dx = 2\left[\frac{1}{3}\sin^3 x\right]_0^{\frac{\pi}{3}} = \frac{2}{3}\left(\frac{\sqrt{3}}{2}\right)^3 = \underline{\frac{\sqrt{3}}{4}}$

(5)





$$V_n = \int_0^{\frac{\pi}{3\cdot 2^n}} \pi\left(2\sin(2^n x)\cos(2^n x) - \sin(2^n x)\right)^2 dx$$

$2^n x = t$ とおく。 $\dfrac{dt}{dx} = 2^n$

x	$0 \to \frac{\pi}{3\cdot 2^n}$
t	$0 \to \frac{\pi}{3}$

$$V_n = \int_0^{\frac{\pi}{3}} \pi\left(2\sin t\cos t - \sin t\right)^2 \cdot \frac{1}{2^n}\,dt$$

$$= \frac{\pi}{2^n}\int_0^{\frac{\pi}{3}} \sin^2 2t - 4\sin^2 t\cos t + \sin^2 t\,dt$$

$$= \frac{\pi}{2^n}\int_0^{\frac{\pi}{3}} \frac{1}{2}(1-\cos 4t) - 4\sin^2 t(\sin t)' + \frac{1}{2}(1-\cos 2t)\,dt$$

$$= \frac{\pi}{2^n}\left[\frac{1}{2}t - \frac{1}{8}\sin 4t - \frac{4}{3}\sin^3 t + \frac{1}{2}t - \frac{1}{4}\sin 2t\right]_0^{\frac{\pi}{3}}$$

$$= \frac{\pi}{2^n}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{16} - \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{8}\right) = \frac{\pi^2}{3\cdot 2^n} - \frac{9\sqrt{3}\pi}{2^{n+4}}$$

$$V_0 = \frac{\pi^2}{3} - \frac{9}{16}\sqrt{3}\pi$$

$\{V_n\}$ は 初項 V_0 公比 $\frac{1}{2}$ の 等比数列.

$$\sum_{n=0}^{\infty} V_n = \left(\frac{\pi^2}{3} - \frac{9}{16}\sqrt{3}\pi\right) \times \frac{1}{1-\frac{1}{2}} = \frac{2}{3}\pi^2 - \frac{9}{8}\sqrt{3}\pi.$$