①

(1) $a_{n+2} = (1-p)a_{n+1} + pa_n$

$\Leftrightarrow \quad a_{n+2} - a_{n+1} = -p(a_{n+1} - a_n)$

$\Leftrightarrow \quad b_{n+1} = -pb_n \quad (n \geqq 1).$

$\therefore \quad b_n = (-p)^{n-1} \times b_1 = (a_2 - a_1)(-p)^{n-1} = (-p)^{n-1}$

b_nの一般項は $\underline{(-p)^{n-1}}$,,

(2) (1)より

$a_{n+1} - a_n = (-p)^{n-1} \quad (n \geqq 1).$

よって $n \geqq 2$のとき.

$$a_n = a_1 + \sum_{k=1}^{n-1}(-p)^{n-1} \quad (n \geqq 2)$$

$$= 1 + \frac{-(-p)^{n-1}+1}{\cancel{+}p+1} = \frac{1}{p+1}\left\{-(-p)^{n-1} + p + 2\right\} \quad (n \geqq 2)$$

ここで上式で $n=1$とすると右辺は

$$\frac{1}{p+1}(-1+p+2) = 1$$

となるので. これは $n=1$においても成立する.

よって

$$\underline{a_n = \frac{1}{p+1}\left\{p+2-(-p)^{n-1}\right\}} \quad (n = 1, 2, 3 \dots)$$

(3) $0 < p < 1$ なので $(-p)^{n-1} \to 0$.

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{p+2-(-p)^{n-1}}{p+1} = \underline{\frac{p+2}{p+1}}$$,,

②

2つのグラフの交点は

$$\frac{3}{2}\tan x = \cos x$$

より　$3\sin x = 2\cos^2 x$

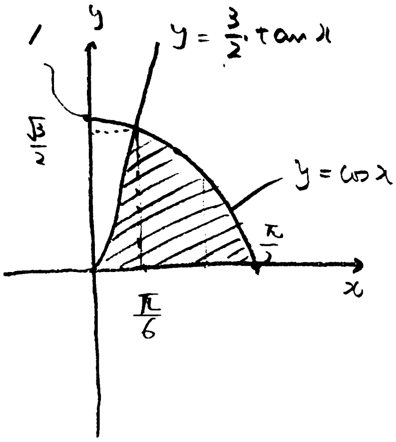
$$3\sin x = 2 - 2\sin^2 x$$

$$2\sin^2 x + 3\sin x - 2 = 0$$

$$(2\sin x - 1)(\sin x + 2) = 0$$

となるので　$\sin x = \frac{1}{2}$

すなわち　$x = \frac{\pi}{6}$ のとき.　$(\because 0 \leqq x < \frac{\pi}{2})$



よって2つのグラフ、および x 軸で囲まれる図形は右上図の斜線部.

したがってもとめる体積を V とすると

$$V = \int_0^{\frac{\pi}{6}} \pi\left(\frac{3}{2}\tan x\right)^2 dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \pi\cos^2 x\, dx$$

$$= \frac{9}{4}\pi\int_0^{\frac{\pi}{6}} \frac{1}{\cos^2 x} - 1\, dx + \frac{\pi}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 1 + \cos 2x\, dx$$

$$= \frac{9}{4}\pi\Big[\tan x - x\Big]_0^{\frac{\pi}{6}} + \frac{\pi}{2}\Big[x + \frac{1}{2}\sin 2x\Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{9}{4}\pi\left(\frac{1}{\sqrt{3}} - \frac{\pi}{6} - 0 + 0\right) + \frac{\pi}{2}\left(\frac{\pi}{2} + 0 - \frac{\pi}{6} - \frac{1}{2}\times\frac{\sqrt{3}}{2}\right)$$

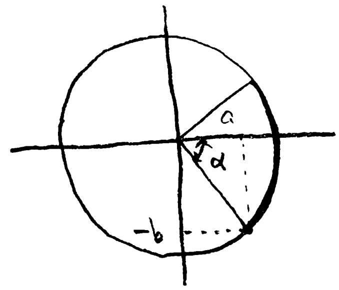
$$= -\frac{3}{8}\pi^2 + \frac{3\sqrt{3}}{4}\pi + \frac{1}{4}\pi^2 - \frac{1}{12}\pi^2 - \frac{\sqrt{3}}{8}\pi$$

$$= \frac{5\sqrt{3}}{8}\pi - \frac{5}{24}\pi^2 = \underline{\frac{5}{8}\pi\left(\sqrt{3} - \frac{1}{3}\pi\right)}$$

③

(1) $a\sin x - b\cos x = \sqrt{a^2+b^2}\ \sin(x-\alpha) = \sin(x-\alpha)$

$\left(ただし\ \ \sin\alpha = \dfrac{b}{\sqrt{a^2+b^2}} = b\ ,\ \cos\alpha = \dfrac{a}{\sqrt{a^2+b^2}} = a \right)$



$$S = \int_0^{\frac{\pi}{2}} |a\sin x - b\cos x|\,dx$$

$$= \int_0^{\frac{\pi}{2}} |\sin(x-\alpha)|\,dx$$

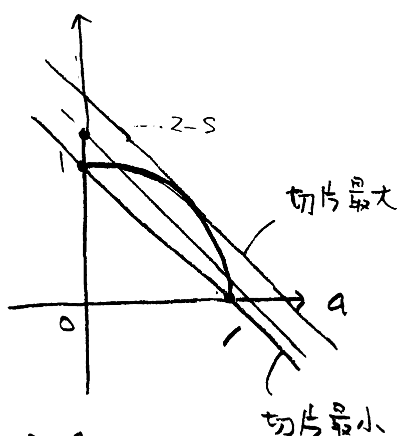
$$= \int_0^{\alpha} -\sin(x-\alpha)\,dx + \int_{\alpha}^{\frac{\pi}{2}} +\sin(x-\alpha)\,dx$$

$$= \int_0^{\alpha} -\sin(x-\alpha)\,dx + \int_{\frac{\pi}{2}}^{\alpha} -\sin(x-\alpha)\,dx$$

$$= \Big[+\cos(x-\alpha)\Big]_0^{\alpha} + \Big[+\cos(x-\alpha)\Big]_{\frac{\pi}{2}}^{\alpha}$$

$$= +1 - \cos\alpha + 1 - \cos\left(\frac{\pi}{2}-\alpha\right)$$

$$= 2 - a - b$$



(2) $S = 2 - a - b$

$\Leftrightarrow\ \ a + b = 2 - S$

これを ab 平面の直線として考えると.

条件は. $a^2+b^2=1\ (a \geqq 0, b \geqq 0)$ と $a+b=2-S$

が交点をもつときの切片の範囲を考えることに等しくなる.

右上図より. 切片は $(a, b) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ のとき最大. で. $\frac{2}{\sqrt{2}}$

$(a, b) = (1, 0)$ または $(0, 1)$ のとき最小. で 1 となる

$$1 \leqq a+b \leqq \frac{2}{\sqrt{2}}$$

$$\Leftrightarrow\ \ 1 \leqq 2-S \leqq \sqrt{2}$$

$$2-\sqrt{2} \leqq S \leqq 1$$

よって S は

$(a, b) = (1, 0)$ または $(0, 1)$ のとき 最大値 1 をとり

$(a, b) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ のとき 最小値 $2-$

④

(1) $w=z$ とすると

$$z=\frac{3(1-i)z-2i}{z+3(1-i)}$$

$$\Leftrightarrow\quad z^2+\cancel{3z}-\cancel{3iz}-\cancel{3z}+\cancel{3iz}+2i=0$$

$$z^2=-2i=2\left(\cos\frac{3}{2}\pi+i\sin\frac{3}{2}\pi\right)$$

$$z=\sqrt{2}\left(\cos\frac{3}{4}\pi+i\sin\frac{3}{4}\pi\right),\ \sqrt{2}\left(\cos\frac{7}{4}\pi+i\sin\frac{7}{4}\pi\right)$$

$$\therefore\quad \underline{z=-1+i,\ 1-i}$$

(2)

$$\frac{w-\beta}{w-\alpha}=\frac{\dfrac{3(1-i)z-2i}{z+3(1-i)}-(1-i)}{\dfrac{3(1-i)z-2i}{z+3(1-i)}-(-1+i)}\qquad(\because \alpha=-1+i,\ \beta=1-i)$$

$$=\frac{3(1-i)z-2i-(1-i)(z+3-3i)}{3(1-i)z-2i+(1-i)(z+3-3i)}$$

$$=\frac{2(1-i)z+4i}{4(1-i)z-8i}$$

$$=\frac{z+\dfrac{2i}{1-i}}{z-\dfrac{2i}{1-i}}\times\frac{1}{2}$$

$$=\frac{1}{2}\times\frac{z+i-1}{z-i+1}=\frac{1}{2}\times\frac{z-\beta}{z-\alpha}$$

$$\therefore\quad \underline{k=\frac{1}{2}}$$

(3) (2) より

$$\frac{z_{n+1}-\beta}{z_{n+1}-\alpha}=\frac{1}{2}\times\frac{z_n-\beta}{z_n-\alpha}\qquad(n=1,2,3\ldots)$$

$\left\{\dfrac{z_n-\beta}{z_n-\alpha}\right\}$ は 初項 $\dfrac{z_1-\beta}{z_1-\alpha}$ すなわち $\dfrac{\beta}{\alpha}$、公比 $\dfrac{1}{2}$ の等比数列

となっている。また $\dfrac{\beta}{\alpha}=\dfrac{1-i}{-1+i}=-1$ なので

$$\frac{z_n-\beta}{z_n-\alpha} = -\left(\frac{1}{2}\right)^{n-1}$$

$$\Leftrightarrow \quad z_n-\beta = -\left(\frac{1}{2}\right)^{n-1}(z_n-\alpha)$$

$$\Leftrightarrow \quad z_n\left(1+\frac{1}{2^{n-1}}\right) = \beta+\left(\frac{1}{2}\right)^{n-1}\alpha$$

$$\Leftrightarrow \quad z_n \times \frac{2^{n-1}+1}{2^{n-1}} = 1-i+\frac{-1+i}{2^{n-1}}$$

$$\Leftrightarrow \quad z_n = \frac{2^{n-1}-2^{n-1}i-1+i}{2^{n-1}+1}$$

$$= \frac{2^{n-1}-1}{2^{n-1}+1}+\frac{1-2^{n-1}}{2^{n-1}+1}i$$

$$= x_n+y_n i$$

$$\therefore \underline{x_n = \frac{2^{n-1}-1}{2^{n-1}+1},\quad y_n = \frac{1-2^{n-1}}{2^{n-1}+1}}$$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\frac{1-\frac{1}{2^{n-1}}}{1+\frac{1}{2^{n-1}}} = 1,\quad \lim_{n\to\infty} y_n = \lim_{n\to\infty}\frac{\frac{1}{2^{n-1}}-1}{1+\frac{1}{2^{n-1}}} = -1.$$

$$\therefore \underline{\lim_{n\to\infty} x_n = 1,\quad \lim_{n\to\infty} y_n = -1}$$